Quantum Advantage for Distributed Symmetry Breaking (Part 3)

Longcheng Li

Joint work with Maxime Flin & Jukka Suomela



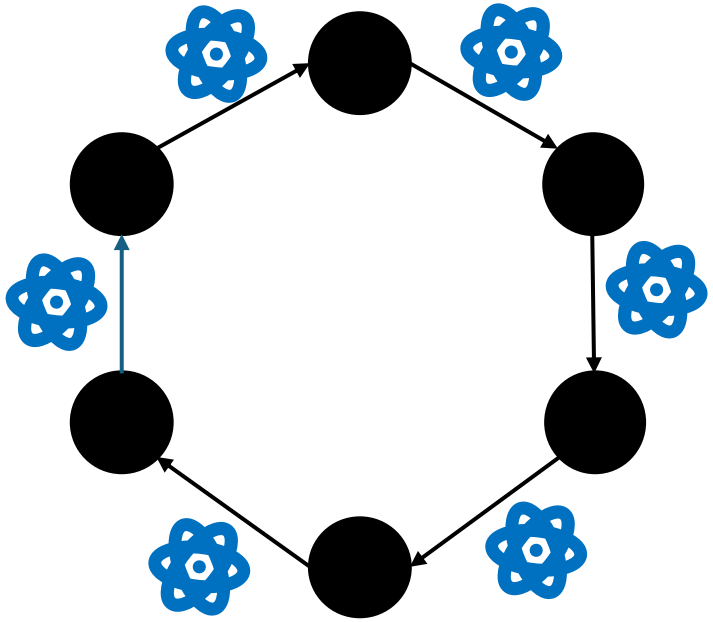
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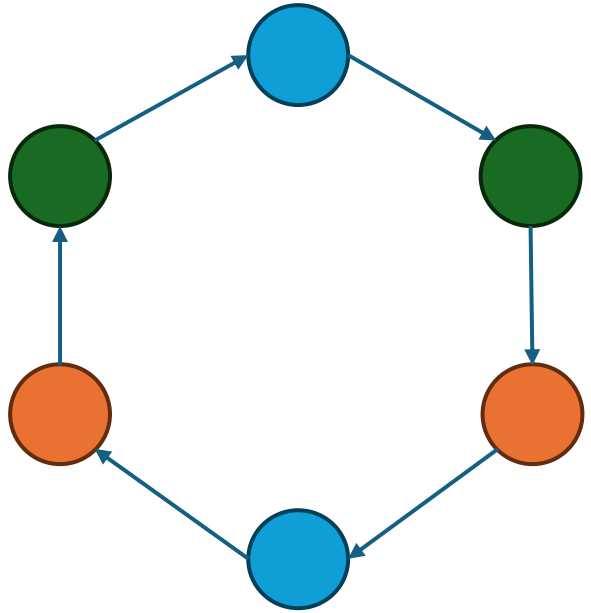
Main theorem

Theorem 1: For $q = 2\times10^4$, there exists a one-way one-round quantum algorithm that q-colors directed cycles whp.



directed n-cycle

One-way one-round
quantum communication



output proper q-coloring
with probability $1 - 1/\mathrm{poly}(n)$

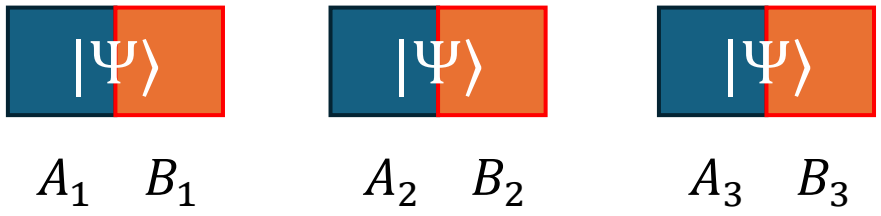
Quantum algorithm

One-way one-round quantum q-coloring algorithm:

- Each node prepares a maximally entangled bipartite state

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle_A |i\rangle_B$$

(N can be arbitrarily large)



Quantum algorithm

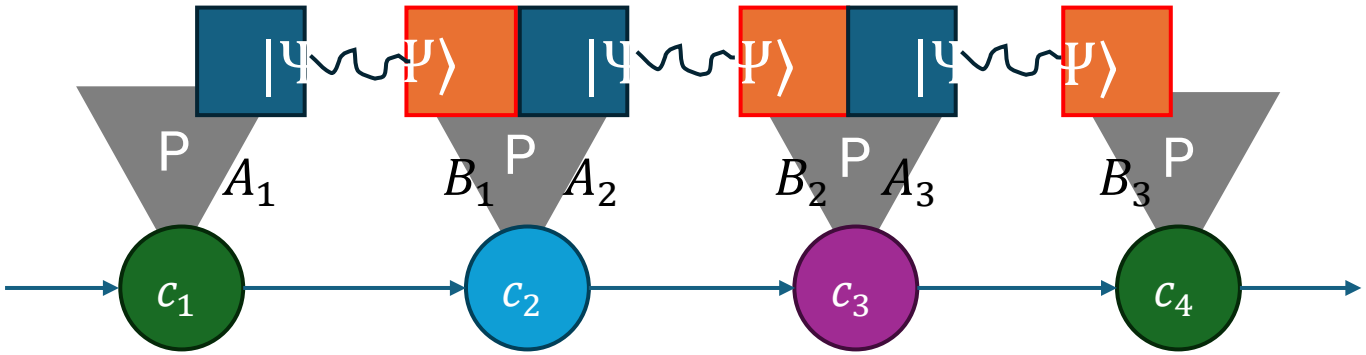
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- Send register B to its successor
- Apply q-outcome measurement $\{P_i\}_{i=1}^{q}$ on its (B, A)



From projection to POVM

Given a quantum state $\rho \in \mathbb{C}^{d \times d}$

Measurement Type	#Outcome	Definition	Probability Formula
Measure by a basis	d	basis $\{\|m_i\rangle\}_{i=1}^{d}$	$\Pr[\text{output } i] = \langle m_i \|\rho\| m_i \rangle$

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POVM	q	$\sum_{i=1}^{q} P_i = I_d$, where P_i are PSDs.	

Naimark's dilation theorem: Any POVM can be realized as projective measurement with extra ancilla states.

This will change the quantum state used in our algorithm.

Quantum algorithm

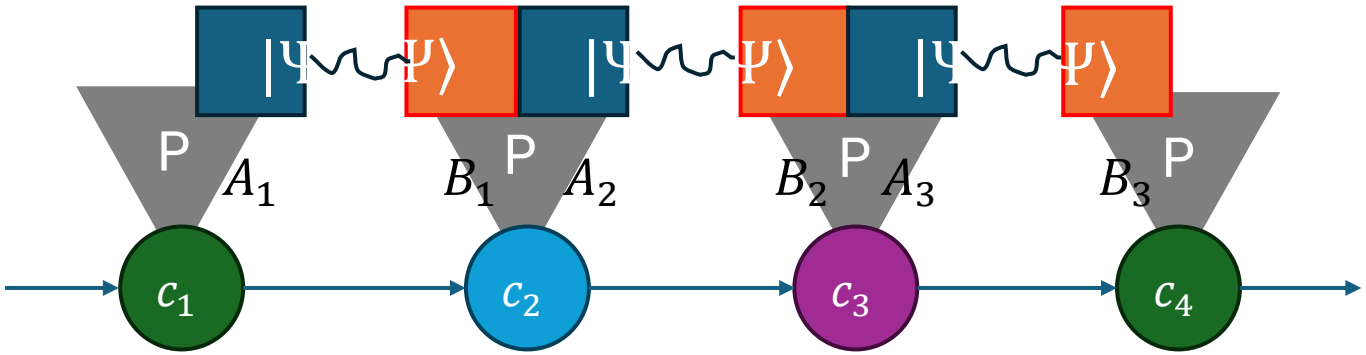
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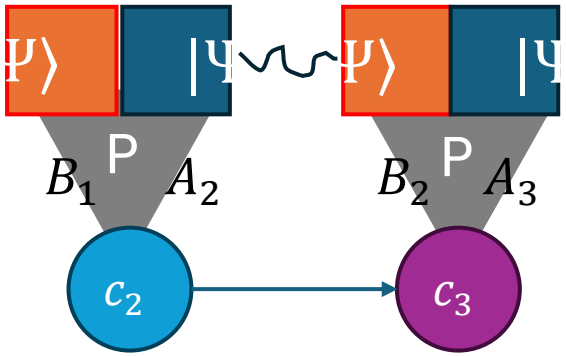
$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle_A |i\rangle_B$$

(N can be arbitrarily large)

- Send register B to its successor
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Computing collision probability



$$\Pr[c_2 = c_3 = a] = \mathrm{Tr}[(P_a \otimes P_a)\ (\ I/N \quad |\Psi\rangle\langle\Psi| \quad I/N\ )]$$

$$= \frac{1}{N^3}\,\mathrm{Tr}[\mathrm{Tr}_A(P_a)^{\mathsf{T}}\mathrm{Tr}_B(P_a)\,]$$

Definition: Given a PSD operator $P \in \mathrm{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$, define its *energy* as $\mathcal{E}(P) := \mathrm{Tr}[\mathrm{Tr}_A(P)^{\mathsf{T}}\mathrm{Tr}_B(P)]$.

$$\Pr[c_2 = c_3] = \frac{1}{N^3}\sum_{a=1}^{q}\mathcal{E}(P_a)$$

⟹ Small total energy = Better coloring algorithm

Why called energy: Projector P

arXiv 2609.09091

$$e_i \otimes e_j \leftrightarrow E_{i,j} \text{ (vectorization)}$$

Since P acts on $\mathbb{C}^N \otimes \mathbb{C}^N \cong \mathbb{C}^{N\times N}$, when it is an orthogonal projector, it projects onto a matrix space $\mathcal{W} \in \mathbb{C}^{N\times N}$.

$$\mathcal{W} \coloneqq \operatorname{span}\{M_1, M_2, \dots, M_{\dim(W)}\}$$

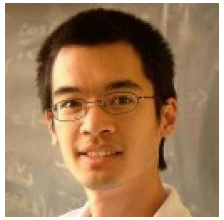
Then

$$\mathcal{E}(P) = \sum_{k,\ell} \|M_k M_\ell\|_F^2 .$$

"Multiplicative energy of space $\mathcal{W}$"

It draws analogy with "additive energy of a set" in additive combinatorics.

I think we settled on "energy" due to the vaguely quadratic nature of the expression, and also because it was monotone.



Terence Tao

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"Multiplicative energy of space $\mathcal{W}$"

Classical setting: $\mathcal{W}$ is spanned by some matrix units $E_{x,y}$

$\|E_{x,y}E_{w,z}\|_F = 1_{y=w}$ ⟹ $x \to y \to z$ forms a directed 2-walk

Thus energy counts 2-walks in a directed graph.

Why called energy: General PSD P

arXiv 2609.09091

Since P acts on $\mathbb{C}^N \otimes \mathbb{C}^N \cong \mathbb{C}^{N\times N}$, write its spectral decomposition

$$P = \sum_k \mathrm{vec}(M_k)\mathrm{vec}(M_k)^\dagger$$

$$\mathrm{vec}(|i\rangle\langle j|) = |i\rangle \otimes |j\rangle$$

$\mathbb{C}^{N\times N}$ $\quad$ $\mathbb{C}^N \otimes \mathbb{C}^N$

Then the same formula works

$$\mathcal{E}(P) = \sum_{k,\ell} \|M_k M_\ell\|_F^2 \,.$$

"Energy of operator P"

Fact: For any PSD P and any such $\{M_k\}$, we have

$$\mathcal{E}(P) = \mathrm{Tr}[\mathrm{Tr}_A(P)^\top \mathrm{Tr}_B(P)] = \textstyle\sum_{k,\ell}\|M_k M_\ell\|_F^2.$$

Finding POVMs with small energy

The problem reduces to:

Fix an integer q. For any $\epsilon > 0$, can we find a dimension N and POVM $\{P_a\}_{a\in[q]}$ acting on $\mathbb{C}^N \otimes \mathbb{C}^N$ such that

$$\frac{1}{N^3}\sum_{a=1}^{q} \mathcal{E}(P_a) \leq \epsilon \text{ ?}$$

$$\Pr[c_i = c_{i+1}]$$

- No for $q \leq 4$ [GL26]
- Yes for $q = 2\times10^4$ (This work)

Technical roadmap

1. Characterize zero-energy operators
2. Recall the infinite-dimensional construction
3. Realize it with finite-dimensional zero-energy PSDs
4. Normalize them into a POVM

Zero energy: projector P

If $P \in \mathrm{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$ is an orthogonal projector, let associated matrix space be

$$\mathcal{W} = \mathrm{span}\{M_1, M_2, \ldots, M_{\dim(\mathcal{W})}\} \subseteq \mathbb{C}^{N \times N}.$$

projector onto $\mathcal{V}$

$\mathcal{E}(P) = 0$ $\Longleftrightarrow$ $\mathcal{W} \subseteq \mathrm{L}(\mathcal{V}^\perp, \mathcal{V})$ for some $\mathcal{V} \oplus \mathcal{V}^\perp = \mathbb{C}^N$ $\overset{\text{vectorize}}{\Longleftrightarrow}$ $P \leq \Pi_{\mathcal{V}} \otimes \Pi_{\mathcal{V}^\perp}^{\mathsf{T}}$

$$\mathcal{E}(P) = \sum_{k,\ell} \|M_k M_\ell\|_F^2 = 0$$

$$M_k M_\ell = 0 \ \ \forall k, \ell$$

$$XY = 0 \ \ \forall X, Y \in \mathcal{W}$$

$\mathrm{L}(\mathcal{V}^\perp, \mathcal{V})$: Linear operators of $\mathcal{V}^\perp \to \mathcal{V}$

$$\begin{array}{c|cc} & \mathcal{V} & \mathcal{V}^\perp \\ \hline \mathcal{V} & 0 & Q \\ \mathcal{V}^\perp & 0 & 0 \end{array} \begin{pmatrix} 0 \\ * \end{pmatrix} = \begin{pmatrix} * \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 0 & Q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & R \\ 0 & 0 \end{bmatrix} = 0$$

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Lemma A: For general PSD P, we similarly have

$\mathcal{E}(P) = 0$ $\Longleftrightarrow$ P is *supported* on $\Pi_{\mathcal{V}} \otimes \Pi_{\mathcal{V}^\perp}^\top$

i.e., $P = (\Pi_{\mathcal{V}} \otimes \Pi_{\mathcal{V}^\perp}^\top) P' (\Pi_{\mathcal{V}} \otimes \Pi_{\mathcal{V}^\perp}^\top)$ for some P'

Recall the infinite construction

Define colors:

an infinite group generated by $s = 10^4$ involutions

- Set $s = 10^4$ and define a group:

$$G = \langle 1,2,\dots,s | i^2 = e \ \forall 1 \le i \le s \rangle.$$

- Set $q = 2s$ colors $\{-1,-2,\dots,-s,+1,+2,\dots,+s\}$.

Associate each color with an element in $\mathbb{C}[G\times G]$:

Complex group algebra of $G\times G$ consisting of

$$\text{finite sum } V = \sum_{g,h} \alpha_{g,h}[g,h]$$

- For each color a $= \pm i$, define "projector"

$$K_a = \frac{[e]\pm[i]}{2} \otimes \frac{[e]\mp[i]}{2} \in \mathbb{C}[G\times G].$$

- The hard part:

Theorem 0: For any $\delta > 0$, we can construct a self-adjoint $W \in \mathbb{C}[G\times G]$ such that

$$\sum_a K_a W^2 K_a \approx_\delta [e,e].$$

Finite-dimensional realization

Goal

For each color $a = \pm i$

Infinite world
$\mathbb{C}[\mathrm{G}\times G]$

$K_a W^2 K_a$



Finite world
$\mathrm{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$

P_a

s.t. 1) $P_a \geq 0$
2) $\mathcal{E}(P_a) = 0$
3) $\sum_a P_a \approx I$

Method

Choose some unitary representation of G
$g \to U_g \in \mathbb{C}^{N\times N}$.

$V := \sum_{g,h} \alpha_{g,h}[g,h]$



$V' := \sum_{g,h} \alpha_{g,h}(U_g \otimes U_h^{\mathsf{T}})$

Finite-dimensional realization

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Method

For each color $a = \pm i$

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Infinite world
$\mathbb{C}[\mathrm{G}\times G]$

$K_a W^2 K_a$

$V \coloneqq \sum_{g,h} \alpha_{g,h}[g,h]$

Finite world
$\mathrm{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$

$P_a \coloneqq K'_a W'^2 K'_a$

$V' \coloneqq \sum_{g,h} \alpha_{g,h}(U_g \otimes U_h^{\mathsf{T}})$

s.t. 1) $P_a \geq 0$ ← everything self-adjoint, so $P_a \coloneqq (W'K'_a)^\dagger (W'K'_a)$

K'_a is a zero-energy projector → 2) $\mathcal{E}(P_a) = 0$

3) $\sum_a P_a \approx I$ ← $\sum_a K_a W^2 K_a \approx [e,e]$ & special property of U_g

(2) $\mathcal{E}(P_a) = 0$

$G = \langle 1,2,\ldots,s | i^2 = e \; \forall 1 \le i \le s \rangle$

Recall that

$$K_a = \frac{[e]+[i]}{2} \otimes \frac{[e]-[i]}{2} \in \mathbb{C}[G \times G]$$

$$[g,h] \to U_g \otimes U_h^\top$$

U_i is some unitary with $U_i^2 = I$

$$K_a' = \frac{I + U_i}{2} \otimes \left(\frac{I - U_i}{2}\right)^\top \in \mathrm{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$$

Orthogonal projector onto the $+1$ eigenspace of U_i

$\mathcal{V}$

Complement orthogonal projector

$\mathcal{V}^\perp$

Lemma A: For PSD P, we have $\mathcal{E}(P) = 0$ iff P is supported on some $\Pi_{\mathcal{V}} \otimes \Pi_{\mathcal{V}^\perp}^\top$

$$K_a' = \Pi_{\mathcal{V}} \otimes \Pi_{\mathcal{V}^\perp}^\top$$

$P_a \coloneqq K_a' W'^2 K_a'$ has zero energy

$$\sum_a K_a W^2 K_a \approx [e,e] \Longrightarrow (3) \sum_a P_a \approx I$$

More precisely: for any $\delta > 0$, we can construct W such that

$$\left\| \sum_a K_a W^2 K_a - [e,e] \right\|_2 \leq \delta \quad \xrightarrow{[g,h] \to U_g \otimes U_h^\top} \quad \left\| \sum_a P_a - I_{N^2} \right\|_? \leq O(\delta)$$

What is the right norm?

For $V := \sum_{g,h} \alpha_{g,h} [g,h]$, define $\|V\|_2 := \sqrt{\sum_{g,h} |\alpha_{g,h}|^2}$

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$$\|[e,e]\|_2 = 1$$

$$[g,h] \to U_g \otimes U_h^{\mathsf{T}}$$

$$\frac{1}{N} \left\| \sum_a P_a - I_{N^2} \right\|_F \leq O(\delta)$$

$$\frac{1}{N} \|I_{N^2}\|_F = 1$$

To preserve 2-norm, it suffices to preserve orthogonality:

For any $g \neq h \in G$, $[g] \perp [h]$ $\Longrightarrow$ $\langle U_g, U_h \rangle := \frac{1}{N} \mathrm{Tr}\left(U_g^\dagger U_h\right) = 1_{g=h}$

Impossible in finite dimension since G is infinite

For $V := \sum_{g,h} \alpha_{g,h} [g,h]$, define $\|V\|_2 := \sqrt{\sum_{g,h} |\alpha_{g,h}|^2}$

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as $N \to \infty$

For $V := \sum_{g,h} \alpha_{g,h} [g,h]$, define $\|V\|_2 := \sqrt{\sum_{g,h} |\alpha_{g,h}|^2}$

Constructing U_g

$$G = \langle 1,2,\dots,s | i^2 = e \; \forall 1 \le i \le s \rangle$$

Lemma B: There exist a sequence of unitary representations

$$g \in G \to U_g^{(N)} \in \mathbb{C}^{N \times N}$$

such that

$$\left\langle U_g^{(N)}, U_h^{(N)} \right\rangle \to 1_{g=h} \quad \text{as} \quad N \to \infty.$$

Proof: Consider $N = 2d$ wlog.

1. Sample Haar random orthogonal matrices $O_1, \dots, O_s \in \mathrm{O}(d)$.
2. For each generator i, Let $U_i := \begin{bmatrix} 0 & O_i^{\mathsf{T}} \\ O_i & 0 \end{bmatrix}$. ⇒ $U_i^2 = I$, so it is a unitary rep.
3. High-dimensional random matrices become orthogonal.
4. By probabilistic method, such deterministic sequence exists.

Almost surely

Finite-dimensional theorem

Theorem 2: Set $q = 2{\times}10^4$. For any $\epsilon > 0$, there exist dimension N and q zero-energy PSD $P_1, \ldots, P_q \in \mathrm{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$ such that

$$\sum_a P_a \approx_\epsilon I_{N^2}.$$

Proof: Given $\epsilon > 0$,

1. Construct $W \in \mathbb{C}[G{\times}G]$ such that $\sum_a K_a W^2 K_a \approx_{\epsilon/2} [e,e]$.
2. Choose large enough N, replace $[g,h]$ with $U_g^{(N)} \otimes U_h^{(N)\mathsf{T}}$ (Lemma B) to get

$$P_a \coloneqq K_a' W'^2 K_a'$$

such that

$$\sum_a P_a \approx_\epsilon I_{N^2}.$$

Final piece: turn P_a into POVMs

Lemma C:

$\sum_a P_a \approx_\epsilon I_{N^2}$ with zero energy



$\sum_a P'_a = I_{N^2}$ with $O(\epsilon N^3)$ energy

Proof: Define $\{P'_a\}$ as pretty good measurement of $\{P_a\}$:

$$P'_a = \sqrt{S^+} P_a \sqrt{S^+} + \frac{1}{q} \Pi_{\ker S} \quad \text{where} \quad S = \sum_a P_a.$$

- When $S = I$, we have $P_a = P'_a$.
- When $S \approx_\epsilon I$, we only perturb energy by a small amount.

Putting things together

Theorem 2: Set $q = 2\times10^4$. For any $\epsilon > 0$, there exist dimension N and q zero-energy PSD $P_1, \dots, P_q \in \mathrm{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$ such that

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⬇ Normalize by Lemma C

Theorem 2': Set $q = 2\times10^4$. For any $\epsilon > 0$, there exist dimension N and q PSD $P_1, \dots, P_q \in \mathrm{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$ such that

$$\sum_a P_a = I_{N^2} \quad \text{and} \quad \frac{1}{N^3}\sum_a \mathcal{E}(P_a) \le \epsilon.$$

⬇ Use $\{P_a\}_{a\in[q]}$ as the measurement

Theorem 1: For $q = 2\times10^4$, there exists a one-way one-round quantum algorithm that q-colors directed cycles whp.