Quantum Advantage for Distributed Symmetry Breaking

Talk 2: The Infinite Construction

Maxime Flin
Aalto University

Longcheng Li
Cambridge University

Jukka Suomela
Aalto University

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The Main Theorem

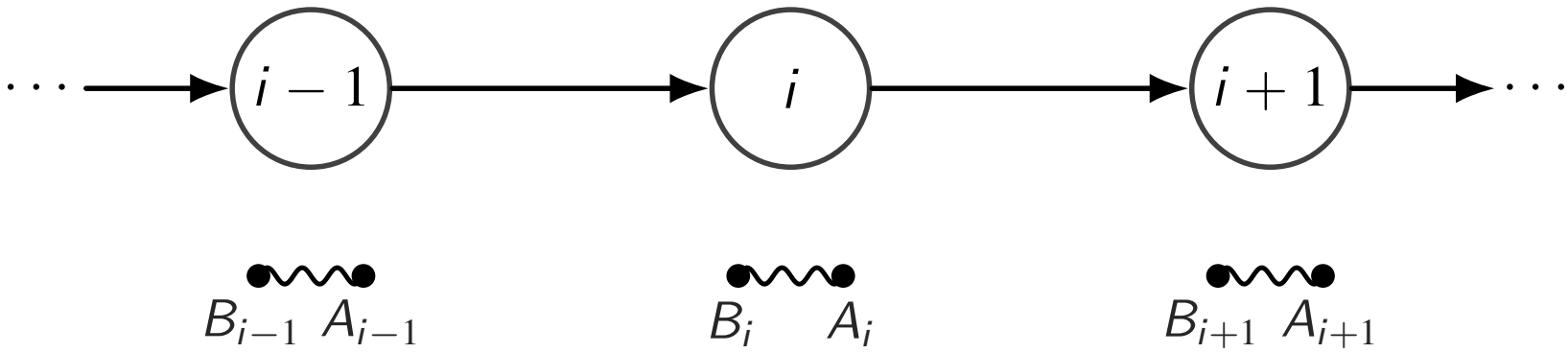
Theorem. For every $\varepsilon > 0$, there is a 1-round one-way quantum-LOCAL algorithm that colors directed cycles with $2 \cdot 10^4$ colors so that

$$\Pr[c_i = c_{i+1}] \leq \varepsilon$$

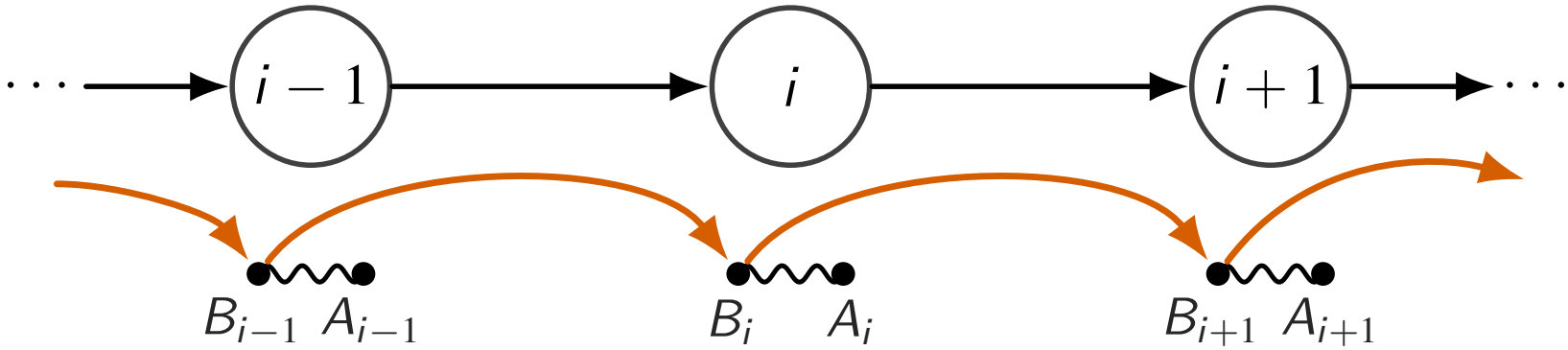
for each $i \in [n]$.

Henceforth $q = 2 \cdot 10^4$ and $s = 10^4$.

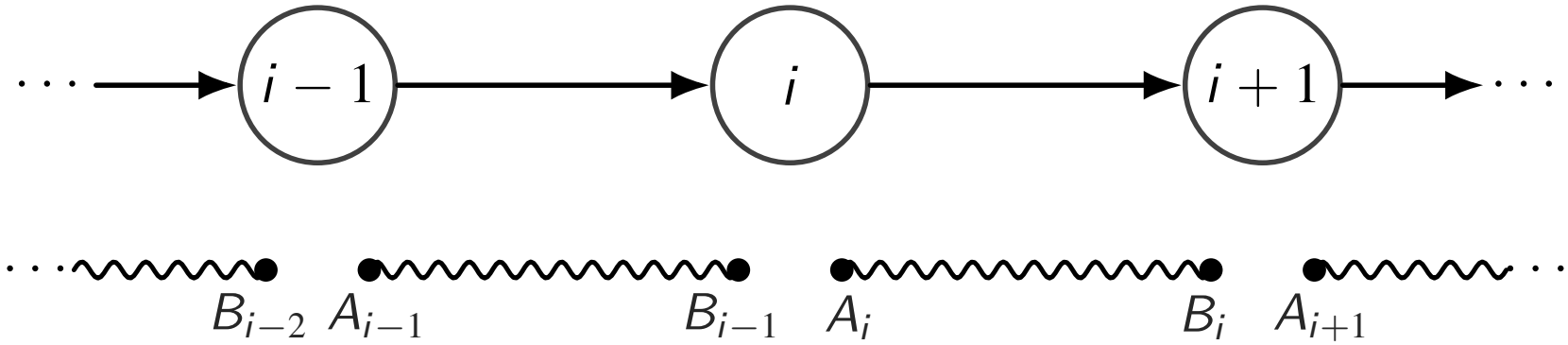
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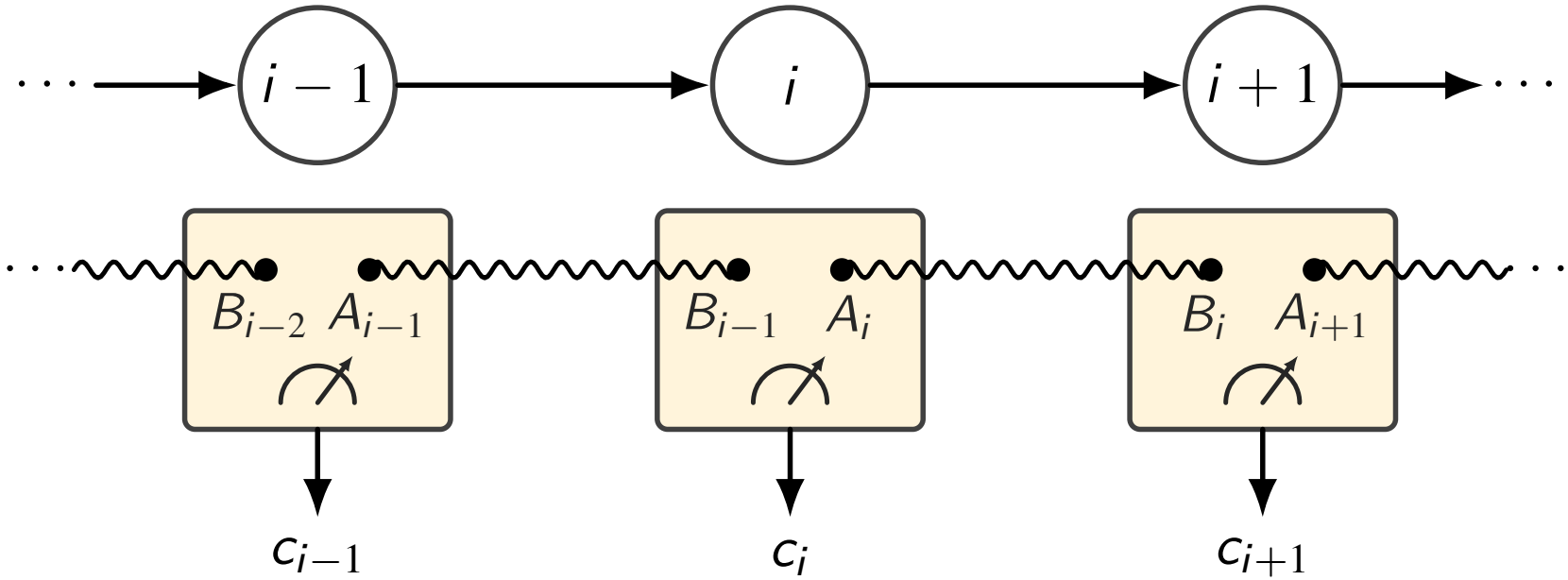
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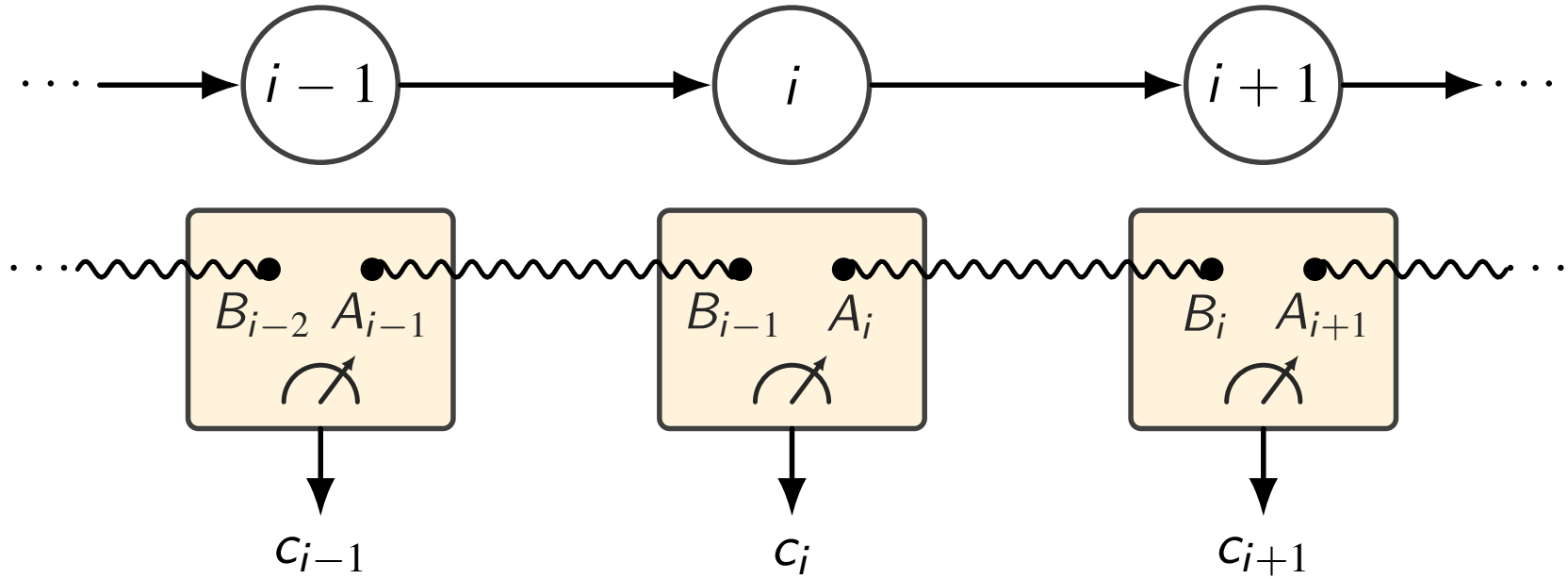
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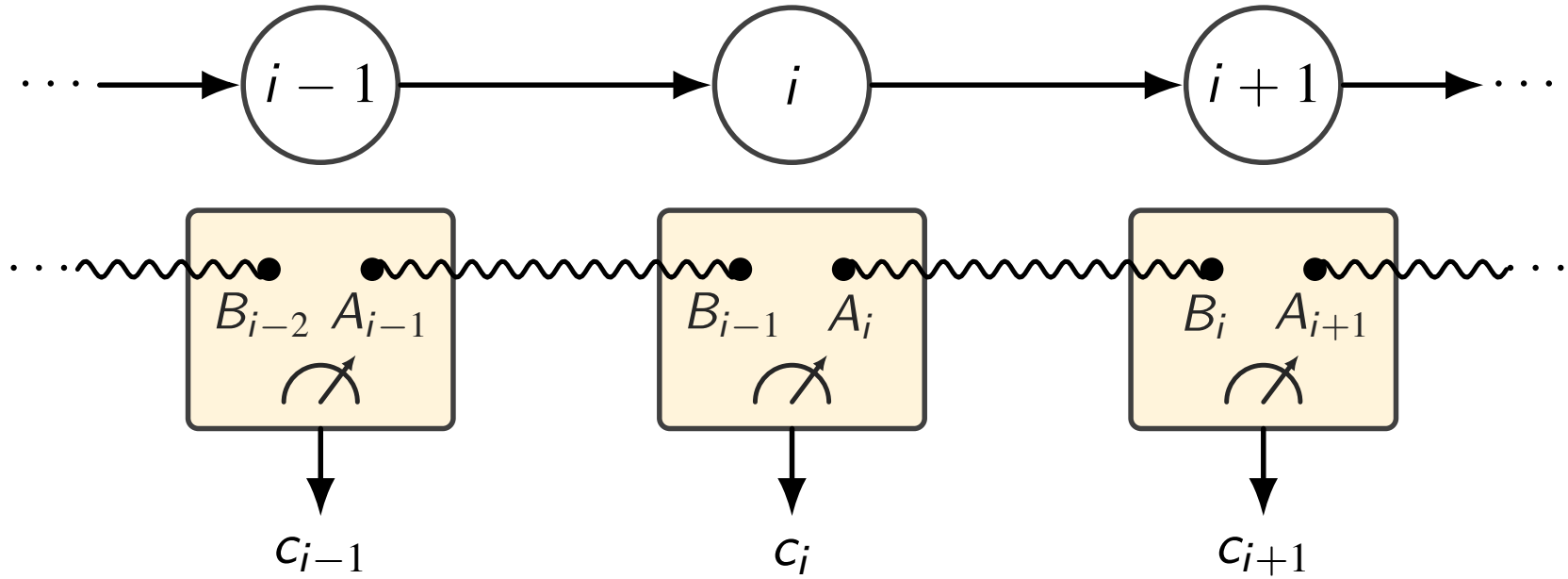


The Algorithm



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The challenge is to construct the measurement

The Infinite Construction

1. operator formulation
2. existence through fixpoint
3. vectorize
4. Taylor approximation

The Finite Construction

1. structure preserving map from infinite to $\mathrm{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$
2. get zero-energy PSD that almost sum to 1
3. renormalize with little energy increase

This Talk: The Infinite Construction

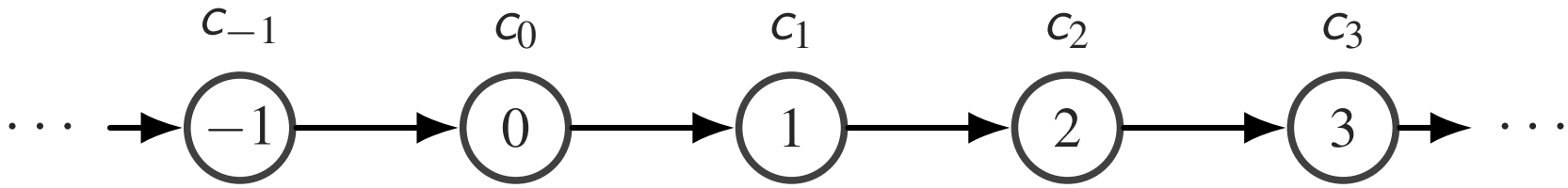
More intuition about the infinite setup (groups?)

Where does the quantum advantage come from?

Some convergence claims

Do not hesitate to interrupt and ask questions!

Recap on 1-dependent colorings of the cycle



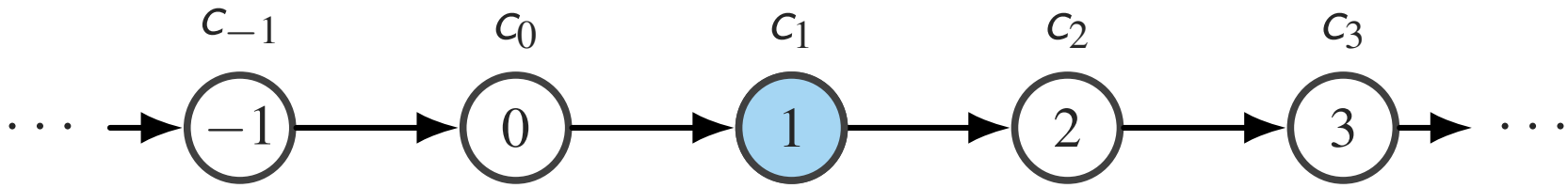
(FD1) always a proper coloring $p(aa) = 0$

(FD2) non-adjacent blocks are independent $p(u?v) = p(u)p(v)$

Holroyd and Liggett proved that 1-dependent 4-colorings exist

We construct a "nice" 1-dependent q-coloring with $q = 2 \cdot 10^4$.

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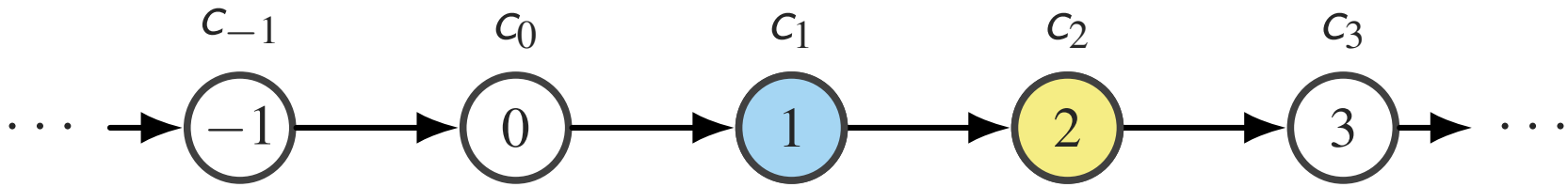
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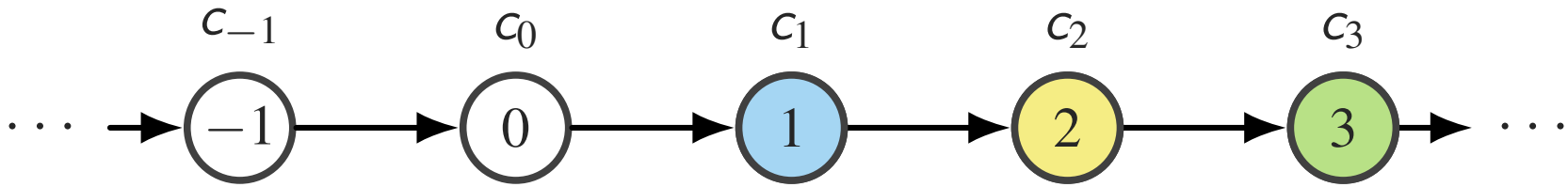
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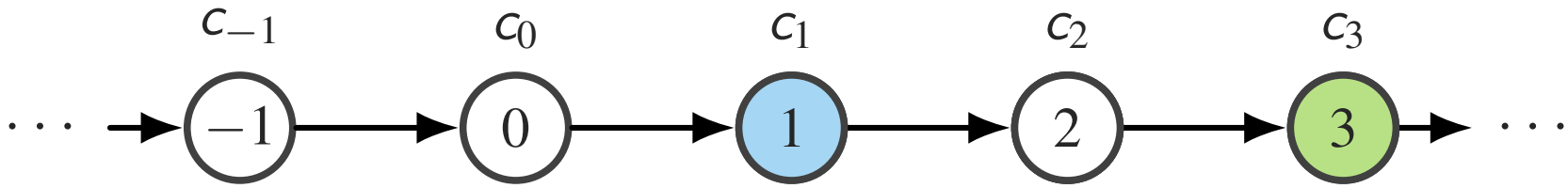
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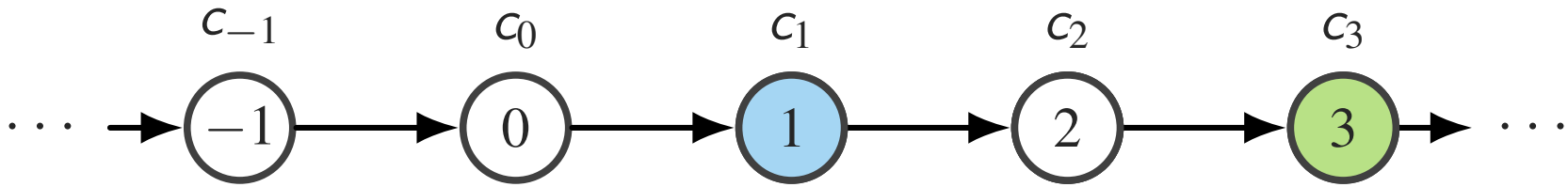
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An Operator Formulation

We want some Hilbert space H and operators

$$T_1, T_2, \ldots, T_q : H \to H$$

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(T1) $T_a^2 = 0$ for all $a \in [q]$.

(T2) $\sum_a T_a = |e\rangle\langle e|$.

(T3) $p(a_1 a_2 \ldots a_k) \geq 0$ for all finite words

Structure Behind $T_a^2 = 0$

Claim.

$$T_a^2 = 0$$

iff

$\exists P_a, Q_a$ orthogonal projections s.t. $T_a = P_a T_a Q_a$ and $P_a Q_a = 0$

Quantum Advantage comes from **non-commutative** projections

$$\{P_a, a \in [q]\} \quad \text{and} \quad \{Q_a, a \in [q]\}\,.$$

What goes wrong classically?

Classical Setup. Each v_i

1. samples $X_i \sim \mu$,
2. sends X_i to v_{i+1},
3. outputs $f(X_{i-1}, X_i) \in [q]$

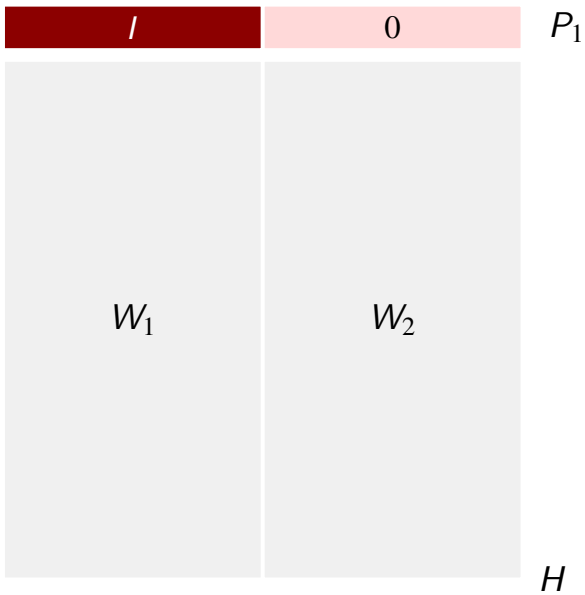
where f is some $(\mu \otimes \mu)$-measurable function.

Operators are over the Hilbert space $H = L^2(\mu)$ and defined as

$$(T_a h)(x) = \mathbb{E}_{Y \sim \mu} [1[f(x, Y) = a] h(Y)]$$

The projections P_a, Q_a that you get are commuting.

What goes wrong classically? The P_a and Q_a commute



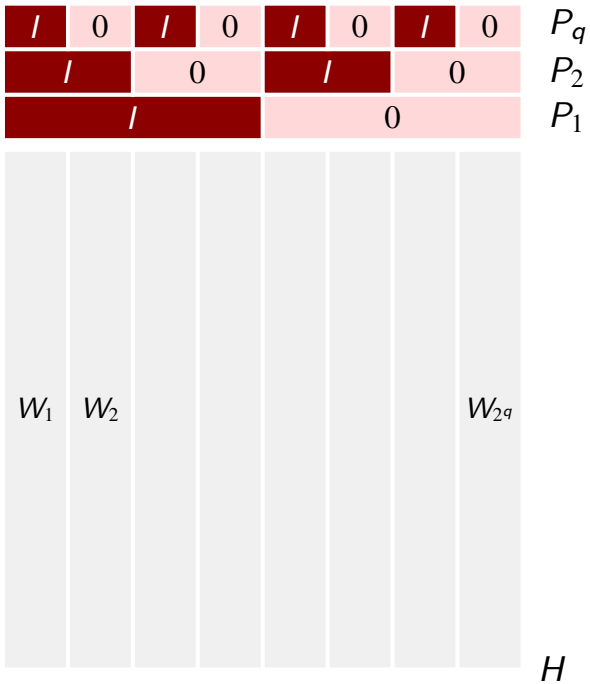
Because they commute, can partition H into

$$W_1 \oplus W_2$$

such that P_1 acts as I on W_1 and as 0 on W_2.

Intuition. Some subspace W_i must have a big probability mass. Then one can contradict (T2).

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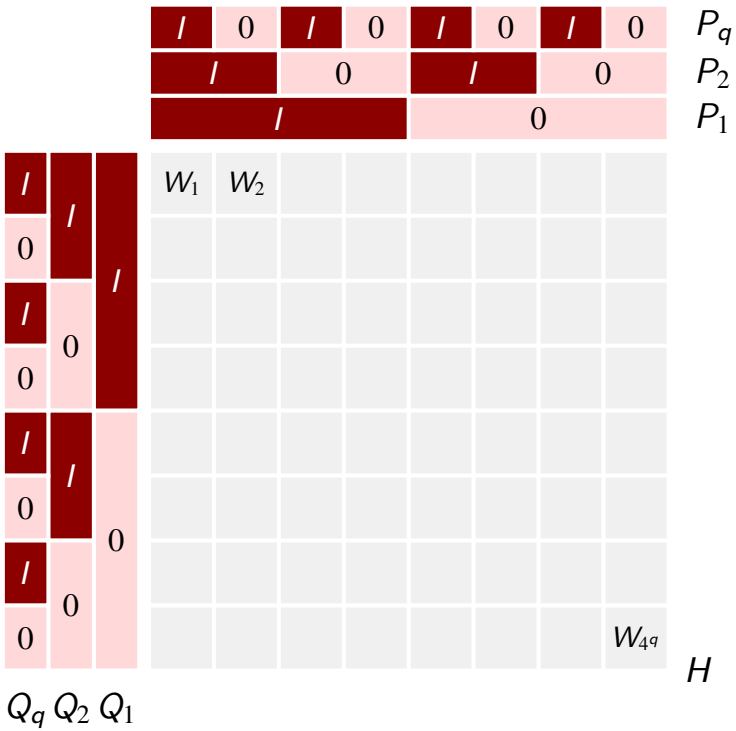
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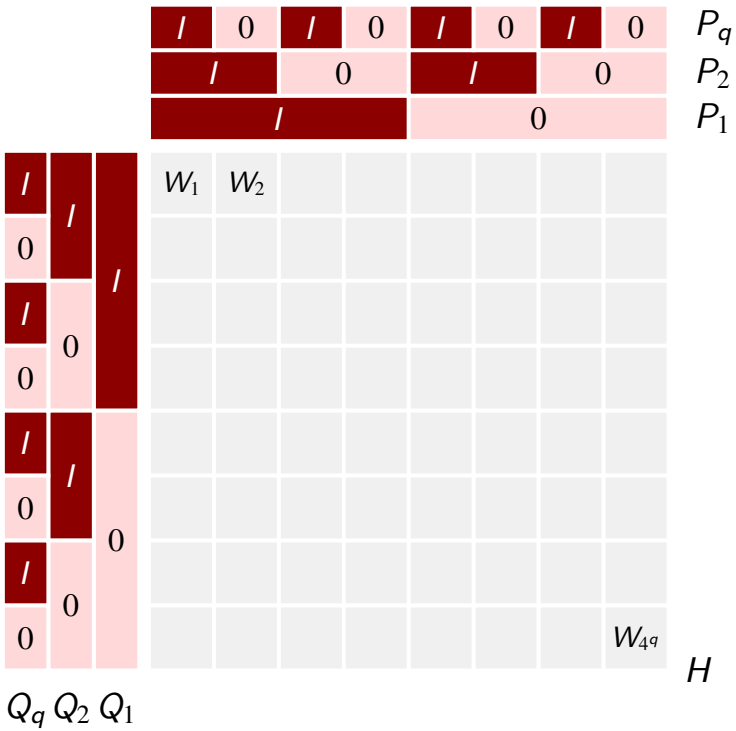
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since $P_a Q_a = 0$ they cannot both act as I on some W_i

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Where do we get non-commutativity?

Consider the group $G = \langle u_1, u_2, \ldots, u_s \mid u_i^2 = e \rangle$

(recall $q = 2 \cdot 10^4$ and $s = q/2$)

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- the group operation is concatenation
- non-abelian group
- each generator is its own inverse

The Hilbert Space

$$H = \ell_2(G) = \left\{ f : G \to \mathbb{C} : \sum_{g \in G} |f(g)|^2 < \infty \right\}$$

For each $g \in G$ let

$$|g\rangle : h \in G \mapsto 1[g = h] .$$

The $\{|g\rangle, g \in G\}$ form a basis and for $i \in [s]$ a generator let

$$L_i|h\rangle = |ih\rangle \quad \text{and} \quad R_i|h\rangle = |hi\rangle .$$

Defining Projections

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For $i \in [s]$ let

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We have

$$P_{+i}P_{-i} = \frac{1}{16}(I - L_i^2)(I - R_i^2) = 0\,.$$

Enforcing T1: $T_a^2 = 0$

Goal. Find some nice operator V and let $T_a = P_a V P_{-a}$.
Proof of T1. $T_a^2 = P_a V P_{-a} P_a V P_{-a} = 0$

Obtaining T2: $\sum_a T_a = |e\rangle\langle e|$

This is how we get 1-dependence.

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This is one linear equation to solve

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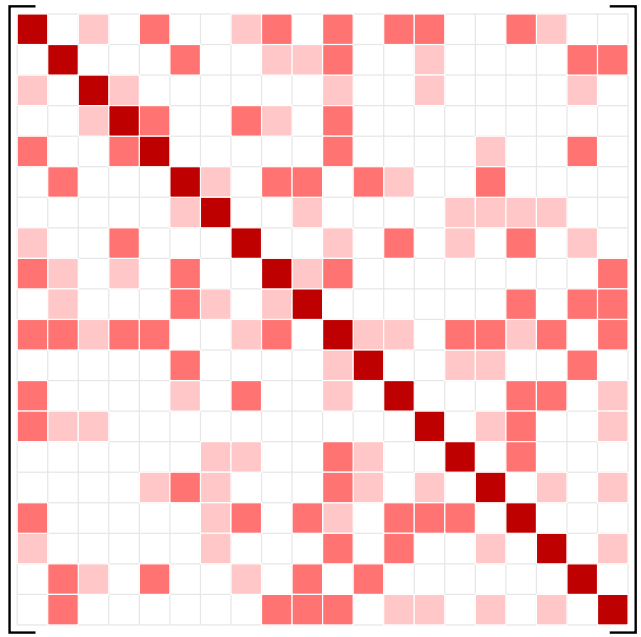
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Contraction Lemma. For all X, we have that

$$\|X - (2/s)S(X)\|_F \leq \lambda \|X\|_F \quad \text{for} \quad \lambda = 3/4 + O(s^{-1/2}) < 1$$

Proving Contraction

Think of S as a matrix with rows and cols indexed by $G \times G$.

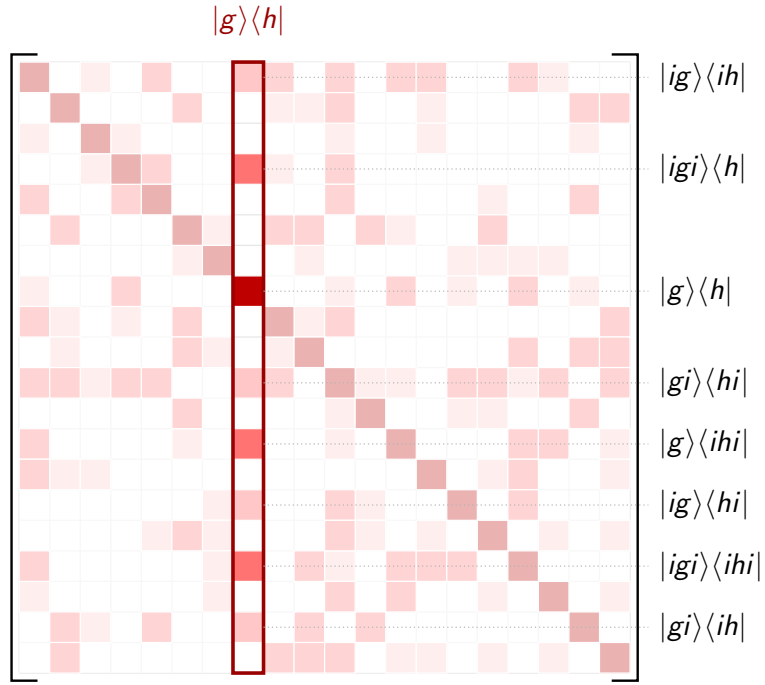


Frobenius norm $=$ Euclidean norm when X is seen as a vector

$$\|X\|_F = \sqrt{\sum_{g,h} |\gamma_{g,h}|^2} \quad \text{where} \quad X = \sum_{g,h} \gamma_{g,h} |g\rangle\langle h|$$

Proving Contraction

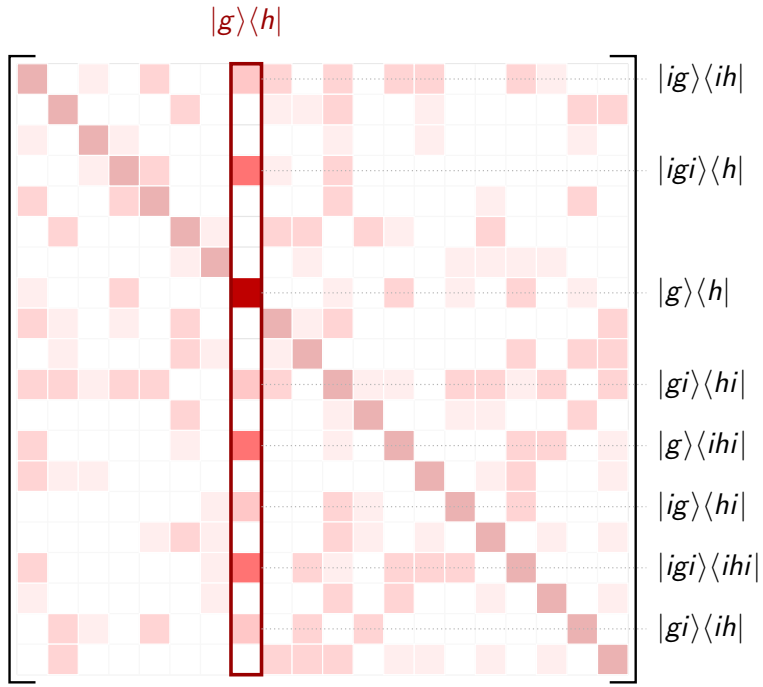
Think of S as a matrix with rows and cols indexed by $G \times G$.



$$
\begin{aligned}
S(|g\rangle\langle h|) = \quad & (s/8)|g\rangle\langle h| \\
& + (1/8)\sum_{i\in[s]} |igi\rangle\langle h| + |g\rangle\langle ihi| + |igi\rangle\langle ihi| \\
& - (1/8)\sum_{i\in[s]} |gi\rangle\langle hi| + |gi\rangle\langle ih| + |ig\rangle\langle hi| + |ig\rangle\langle ih|
\end{aligned}
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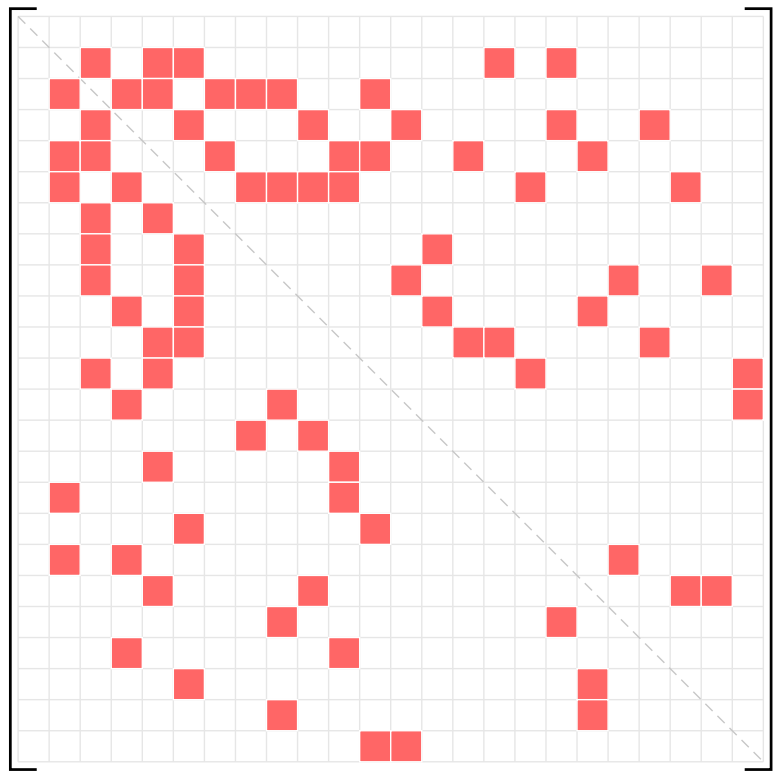
$$S(X) = D(X) + (1/8)\sum_{i=1}^{3} F_j(X) - (1/8)\sum_{i=4}^{4} F_j(X)$$

Idea. Split the contribution of the diagonal and off-diagonal terms:

$$\|X - (2/s)S(X)\|_F \leq \underbrace{\|X - (2/s)D(X)\|_F}_{\leq (3/4)\|X\|_F} + \frac{1}{4s}\sum_j \|F_j(X)\|_F$$

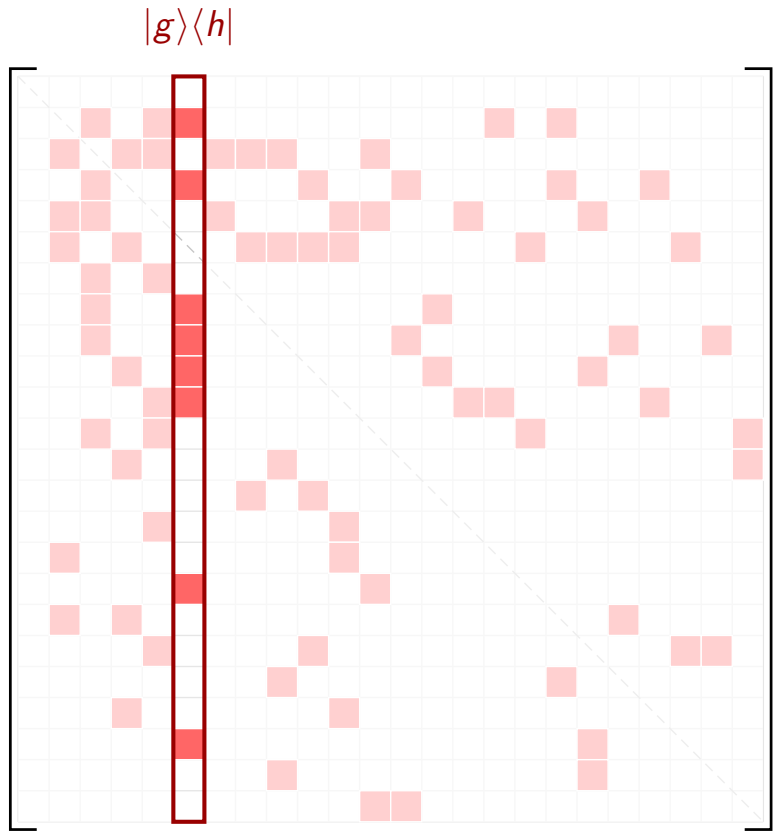
Off-Diagonal Terms

$$F_1(|g\rangle\langle h|) = \sum_{i:igi\neq g} |igi\rangle\langle h|$$



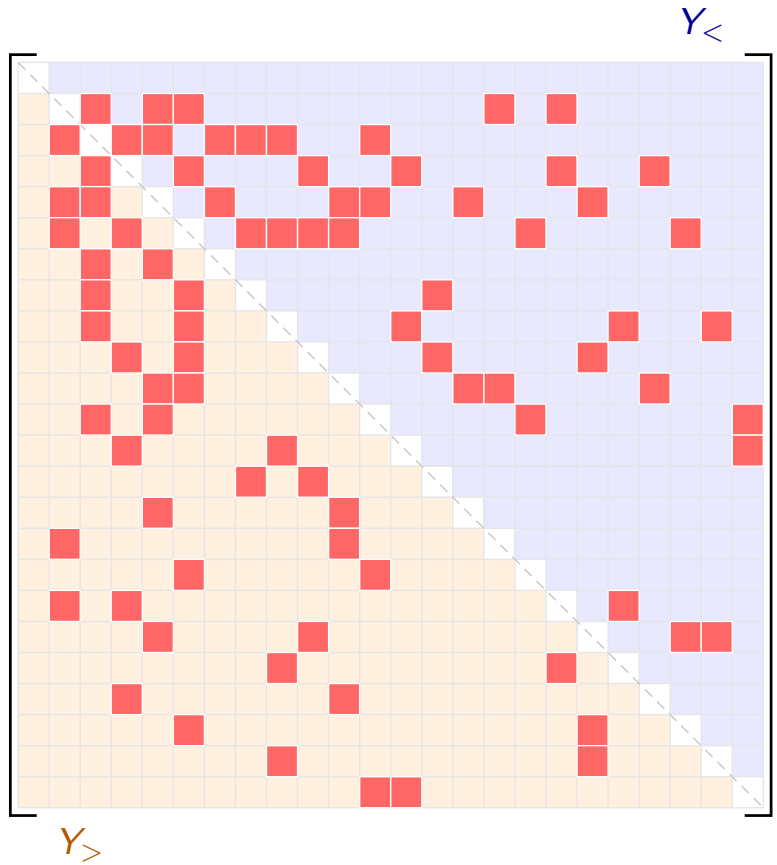
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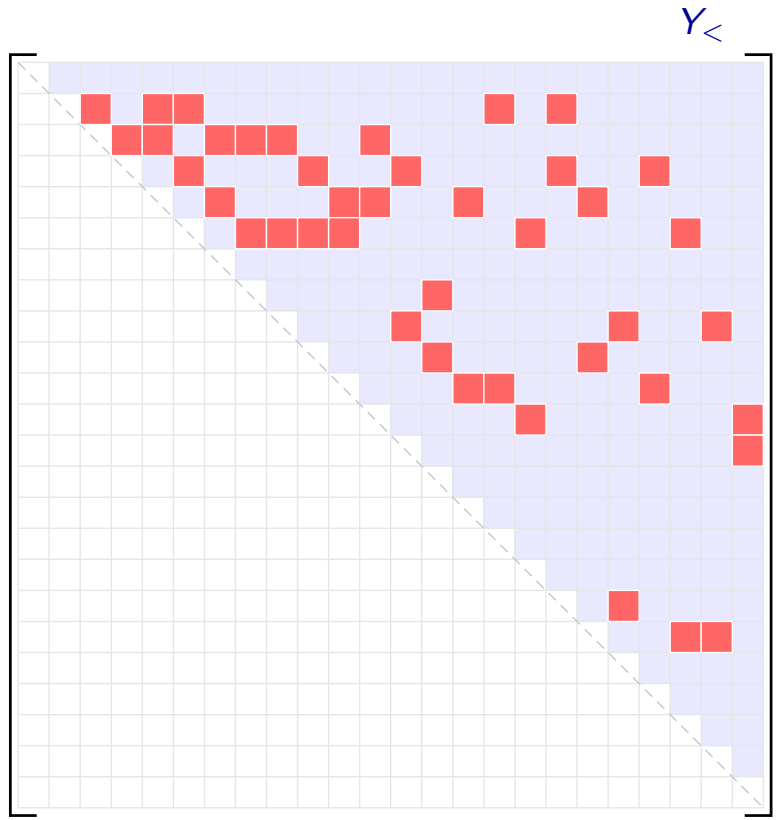
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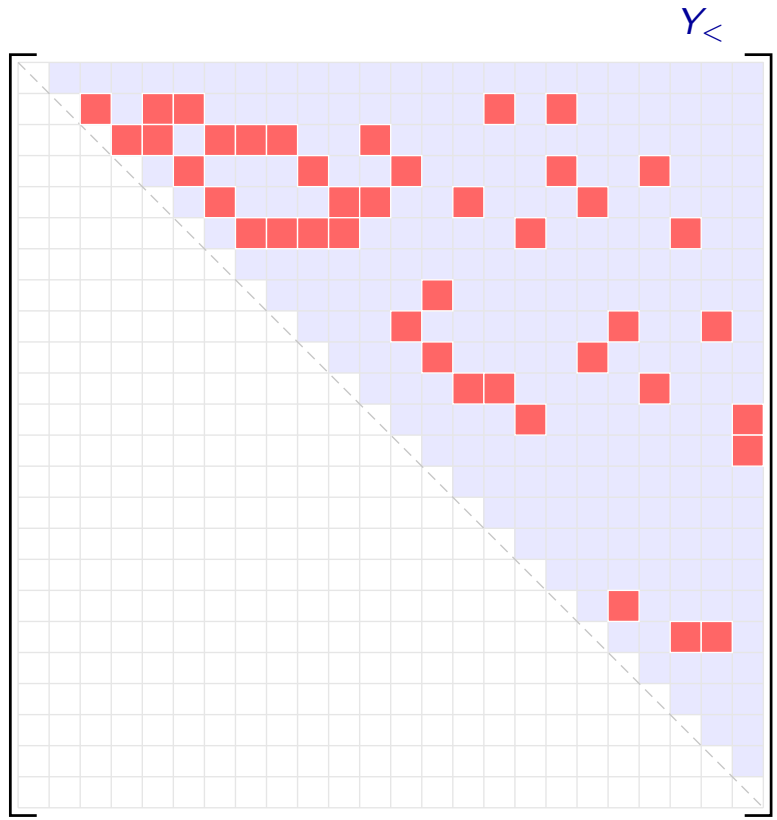
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Off-Diagonal Terms

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Claim. $\|F_1(X)\|_F \leq 2\sqrt{2s}\|X\|_F$.

Proving Contraction Lemma

Step 1. Split diagonal and off-diagonal contributions

$$S(X) = D(X) + (1/8)\sum_{j=1}^{3} F_j(X) - (1/8)\sum_{j=4}^{7} F_j(X)$$

Step 2. Diagonal contributions are easy to bound

$$\|X - (2/s)D(X)\|_F \leq (3/4)\|X\|_F$$

Step 3. Off-diagonal contributions: $\|F_j(X)\|_F \leq 2\sqrt{2s}\|X\|_F$.

Conclude.

$$\begin{aligned}\|X - (2/s)S(X)\|_F &\leq (3/4)\|X\|_F + \frac{1}{4s}\sum_j \|F_j(X)\|_F \\ &\leq \left(\frac{3}{4} + \frac{7}{\sqrt{2s}}\right)\|X\|_F\end{aligned}$$

Vectorization

We have an operator

$$V_n = \sum_{g,h \in G} \gamma_{g,h} |g\rangle\langle h| \in \mathrm{L}(H)$$

where H is an infinite dim Hilbert space and only finitely many $\gamma_{g,h} \neq 0$.

Interpret the operator as a polynomial

$$V_n' = \sum_{g,h \in G} \gamma_{g,h} X_{g,h} \in \mathbb{C}[G \times G]$$

with infinitely many indeterminates $X_{g,h}$ for $g, h \in G$ such that $X_{g,h} X_{g',h'} = X_{gg',hh'}$

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$$V_n' = \sum_{g,h \in G} \gamma_{g,h} [g,h] \in \mathbb{C}[G \times G]$$

with infinitely many indeterminates $[g,h]$ for $g,h \in G$ such that $[g,h][g',h'] = [gg',hh']$

we want an operator in $\mathrm{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$

T1/T2/T3 in $\mathbb{C}[G \times G]$?

(T1) The projections P_a become polynomials K_a

(T2) The norm does not change

(T3) The multiplicative structure changed, so we need a different guarantee

$$\text{for } X, Y \in L(H), \qquad X' \cdot Y' = (\ \underbrace{X * Y}_{\text{convolution}}\ )'$$

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If $V_n' = W^\dagger W$ (wrt multiplication in $\mathbb{C}[G \times G]$) then the "corresponding finite matrix" is always PSD

Can we find some $W = W^\dagger$ such that $W^2 \approx V_n'$? And use $K_a W^2 K_a$

Taylor Approximation

$(s/2)V_n' = I + R_n$ where $\|R_n\| = O(s^{-1/2})$.

Use

$$\sqrt{I + R_n} = I + R_n/2 - R_n^2/8 + R_n^3/16 + \dots$$

Take $X = I + R_n/2 - R_n^2/8 + \dots$ a finite truncation so that

$$\sum_a K_a W^2 K_a \approx I \quad \text{where} \quad W = \sqrt{\frac{2}{s}} X$$

Taylor Convergence

For the Taylor series to converge, need a universal constant C st

$$\|R_n\| < 1/C \quad \text{and} \quad \underbrace{\|A \cdot B\|}_{\text{mult. in } \mathbb{C}[G \times G]} \leq C\|A\|\|B\|$$

The Frobenius norm does **not** have this property

For instance: $X = [e, e] + [12, e] + [1212, e] + \ldots + [(12)^k, e]$.

$$X^2 = \sum_{i,j=0}^{k} [(12)^{i+j}, e] = \sum_{\ell=0}^{2k} |\{(i,j) : i + j = \ell\}|[(12)^\ell, e]$$

Hence $\|X\|_F^2 = k + 1$ but
$\|XX\|_F^2 = 1 + 2^2 + 3^2 + \ldots + (k+1)^2 + \ldots = \Theta(k^3)$.

Changing the norm

Denote by $X_{k,\ell}$ the part of X supported on $[g,h]$ with $|g| = k$ and $|h| = \ell$:

$$X_{k,\ell} = \sum_{|g|=k,|h|=\ell} \gamma_{g,h}[g,h]$$

Then one can show[1]

$$\|X_{k,\ell} Y_{p,q}\|_w \leq 4 \cdot 2^{(k+\ell)/2} 2^{(p+q)/2} \|X_{k,\ell}\|_F \|Y_{p,q}\|_F$$

The Weighted Norm

$$\|X\|_w = \sum_{k,\ell \geq 0} 2^{(k+\ell)/2} \|X_{k,\ell}\|_F$$

one can show $\|XY\|_w \leq 4\|X\|_w \|Y\|_w$.

[1] Based on a similar result by Haagerup from 1978 about free groups

Contraction Lemma Still Works with worse constants

Contraction Lemma. For all X, we have that

$$\|X - (2/s)S(X)\|_w \leq \lambda \|X\|_w \quad \text{for} \quad \lambda = 3/4 + O(s^{-1/2}) < 1$$

Because

$$\begin{aligned} S(|g\rangle\langle h|) = \quad & (s/8)|g\rangle\langle h| \\ & + (1/8)\sum_{i\in[s]} |igi\rangle\langle h| + |g\rangle\langle ihi| + |igi\rangle\langle ihi| \\ & - (1/8)\sum_{i\in[s]} |gi\rangle\langle hi| + |gi\rangle\langle ih| + |ig\rangle\langle hi| + |ig\rangle\langle ih| \end{aligned}$$

is supported on words of length $|g| + |h| \pm O(1)$.

Concluding with Taylor

Let $C_n = (s/2)V_n - [e, e]$.
Need to argue that $\|C_n\|_w < 1/4$ to get convergence.

Using Contraction Lemma,

$$\begin{aligned}\|C_{n+1}\|_w &\leq \lambda\|C_n\|_w + \|[e, e] - (2/s)S'([e, e])\|_w \\ &\leq \lambda\|C_n\|_w + O(s^{-1/2}) \\ &\leq O(s^{-1/2})\sum_j \lambda^j < 1/4\end{aligned}$$

for s a large enough constant.

Summary

Theorem. For all $\delta > 0$, there is a self-adjoint $W \in \mathbb{C}[G \times G]$ s.t.

$$\left\| \sum_a K_a W^2 K_a - I \right\|_F \leq \delta$$

Proof Sketch.

1. Get $\sum_a P_a V_n P_{-a} \approx |e\rangle\langle e|$
2. Vectorize from operators to group algebra
3. Get W such that $W^2 \approx V_n$ using Taylor approximation

Remarks

- The P_a and K_a projections ensure the coloring is proper
- Using W^2 ensures positivity

Next Week: Longcheng's Talk!

The Finite Construction

1. structure preserving map from infinite to $\mathrm{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$
2. get zero-energy PSD that almost sum to 1
3. renormalize with little energy increase

Why does this give *quantum measurements*?

The distance to I is bounded only in *normalized Frobenius norm*, how do we normalize that?

Longcheng is a *true* quantum person, he will give a quantum information perspective on this!