Quantum Advantage for Distributed Symmetry Breaking

Talk 1: The High Level Picture

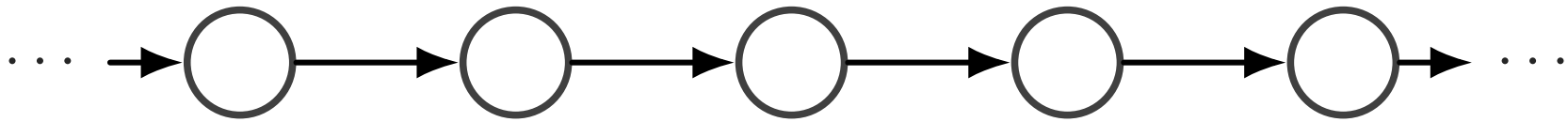
Maxime Flin
Aalto University

Longcheng Li
Cambridge University

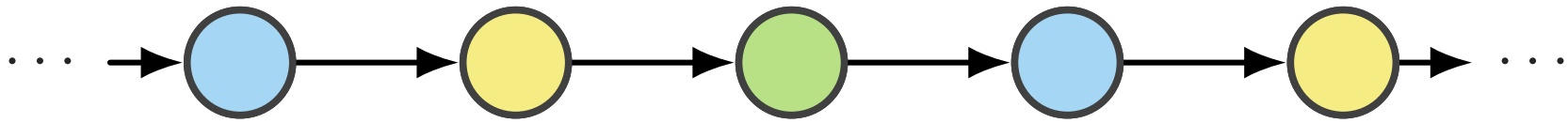
Jukka Suomela
Aalto University

7 October 2026

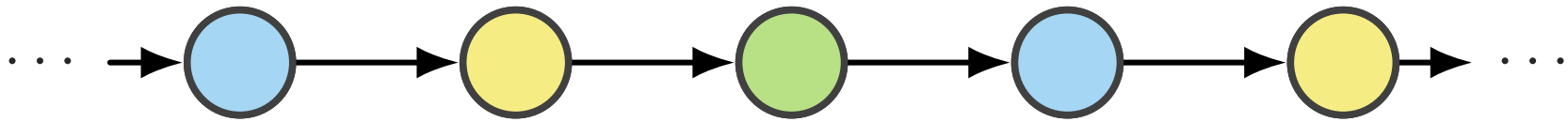
Coloring Directed Cycles



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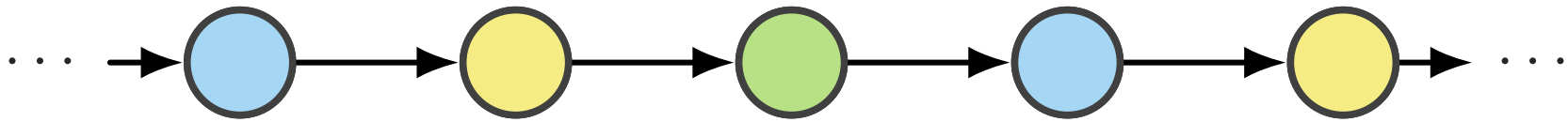


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Quantum Lower Bounds.

- 3-coloring whp. by 1-round 1-way algorithm is impossible
- 4-coloring whp. by 1-round 1-way algorithm is impossible (Gur-Li, 2026)
- q-coloring wp 1 requires $\Omega(n)$ rounds (CR**F**dGRS26, 2026)

Results and Corollaries

New! There is a quantum-LOCAL algorithm that 3-colors directed cycles in $O(1)$ rounds whp.

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- $(\Delta + 1)$-coloring,
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- $(\Delta + 1)$-coloring,
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- maximal matching, and
- maximal independent set.

Corollary! The complexity of 3-coloring rooted trees in quantum-LOCAL is $\Theta(\log^* \Delta)$.

The Main Theorem

Theorem. For every $\varepsilon > 0$, there is a 1-round one-way quantum-LOCAL algorithm that colors directed cycles with $2 \cdot 10^4$ colors so that

$$\Pr[c_i = c_{i+1}] \leq \varepsilon$$

for each $i \in [n]$.

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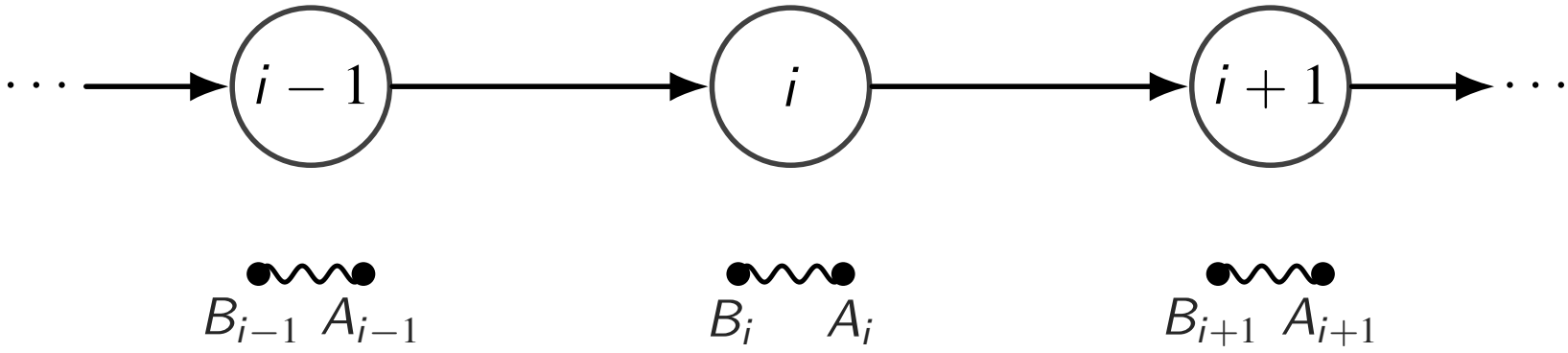
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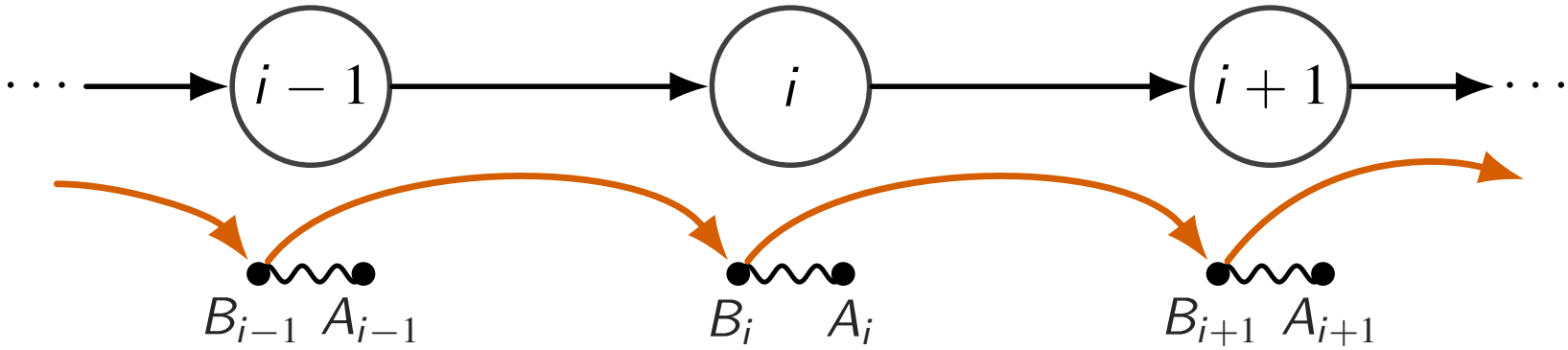
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- Seems like the key human input was Gur-Li 2026

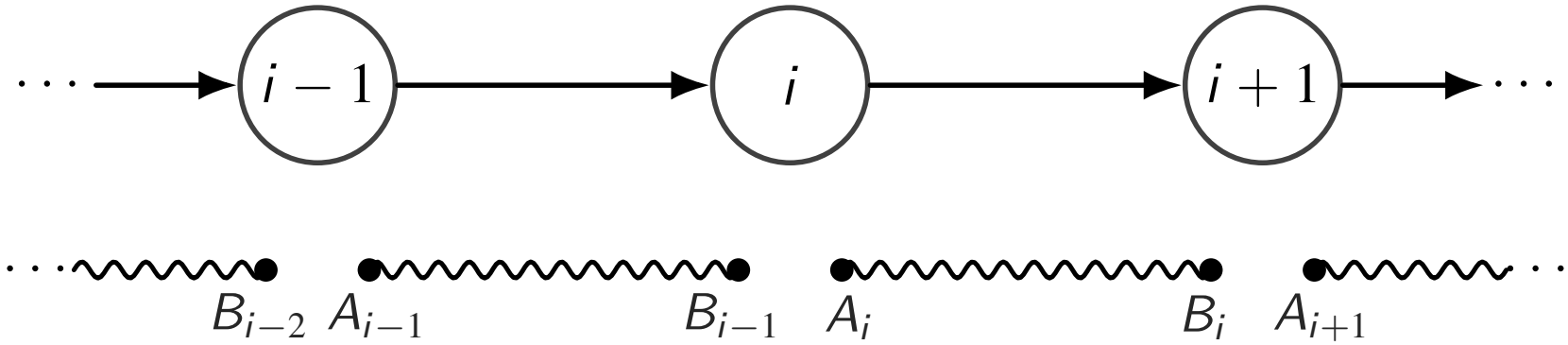
The Algorithm



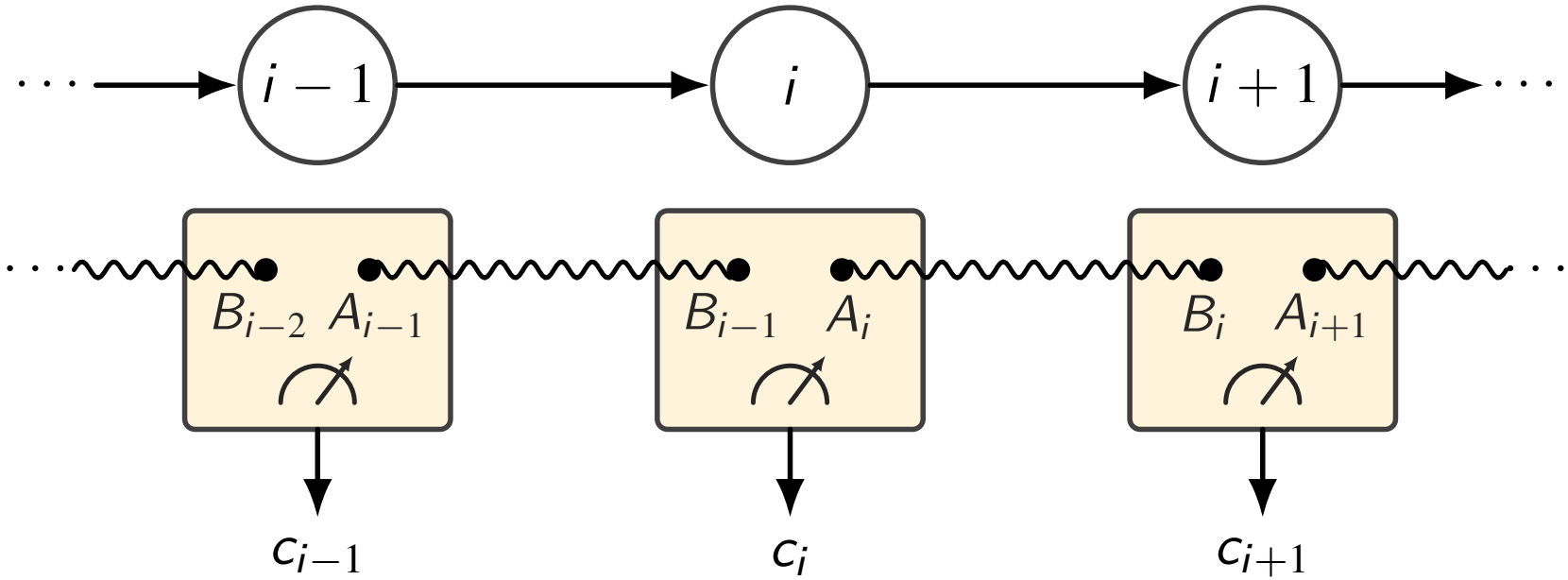
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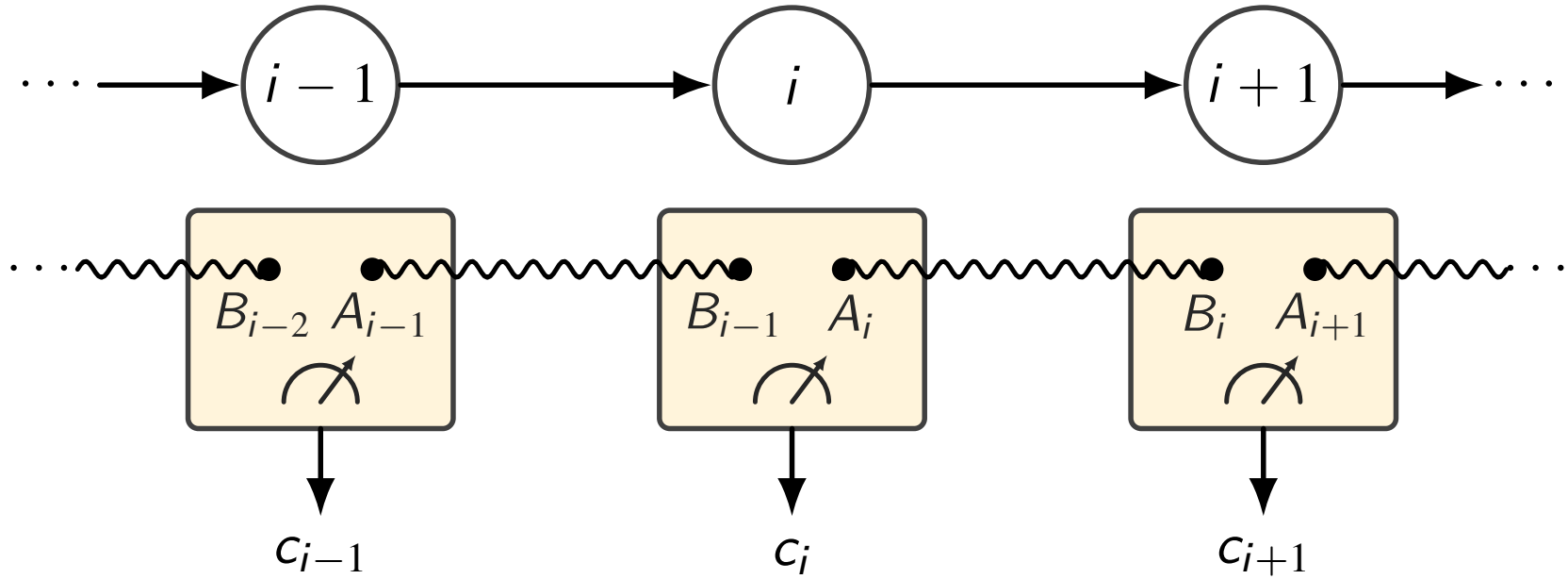
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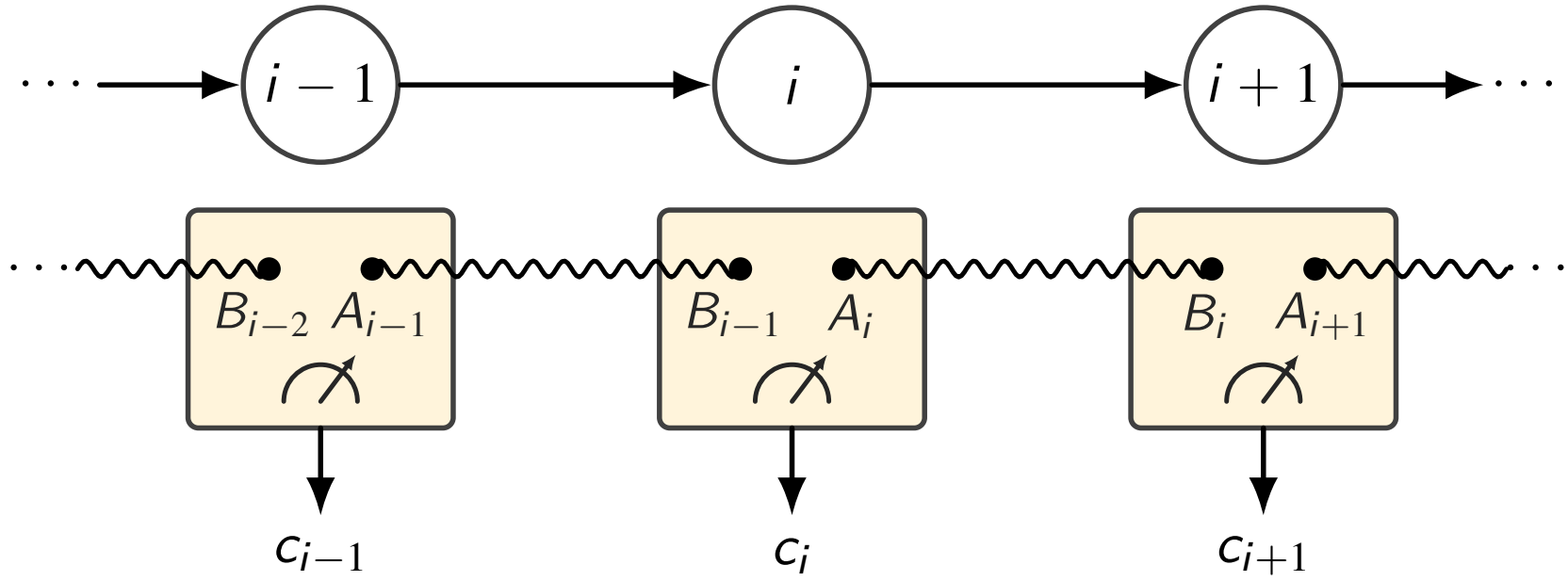


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The challenge is to construct the measurement

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2. existence through fixpoint
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The Finite Construction

1. structure preserving map from infinite to $\mathrm{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$
2. get zero-energy PSD that almost sum to 1
3. renormalize with little energy increase

This Talk

How could someone have discovered this?

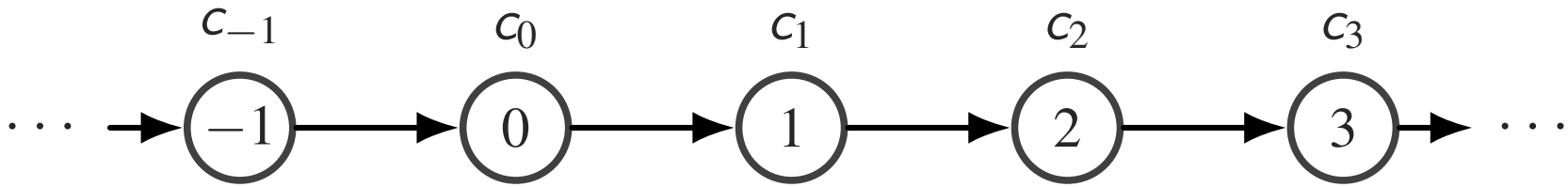
Is this a general scheme we can use for other problems?

Do not hesitate to interrupt and ask questions!

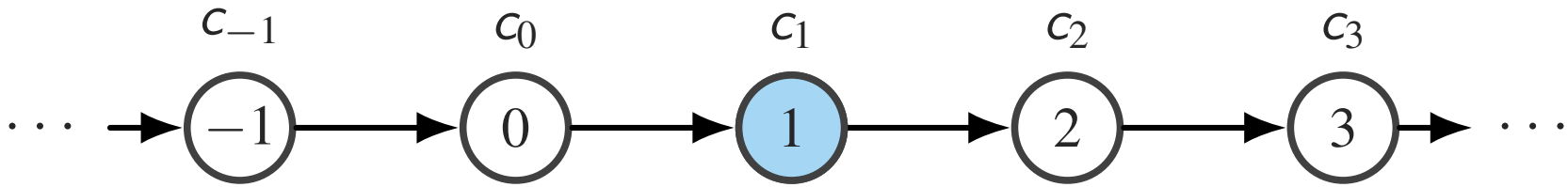
Part I:

The Infinite Construction

1-dependent colorings of the cycle

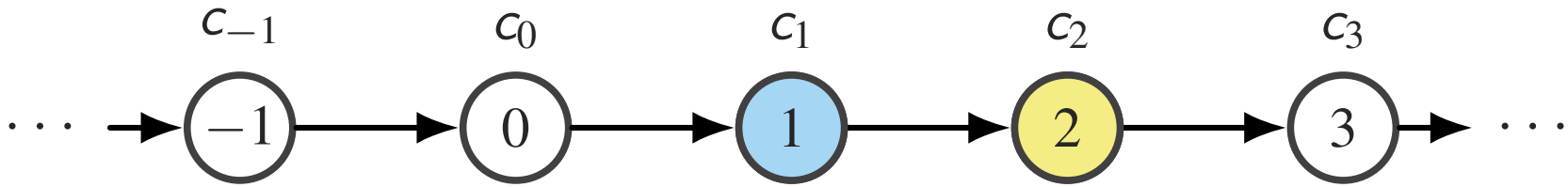


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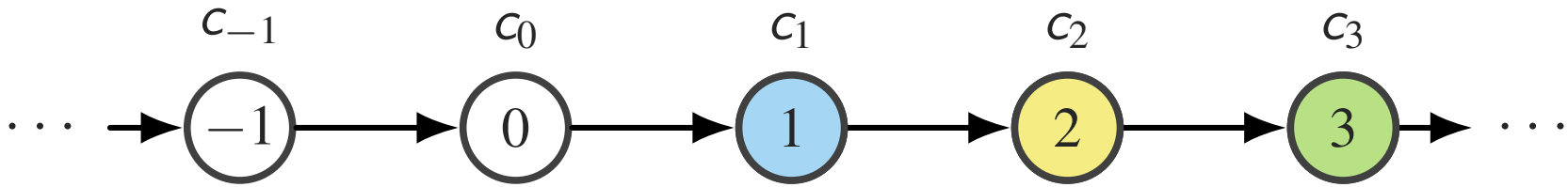
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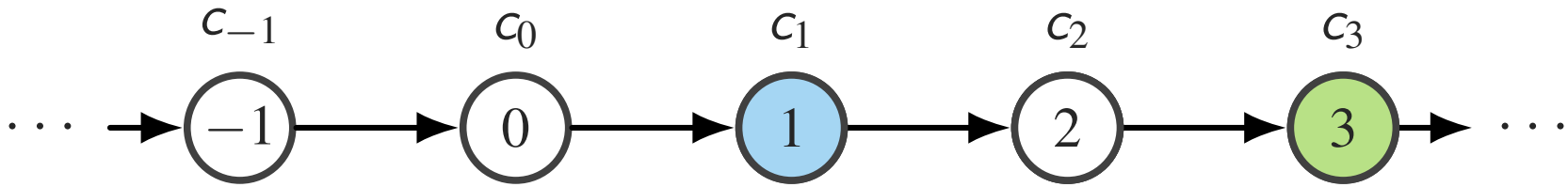
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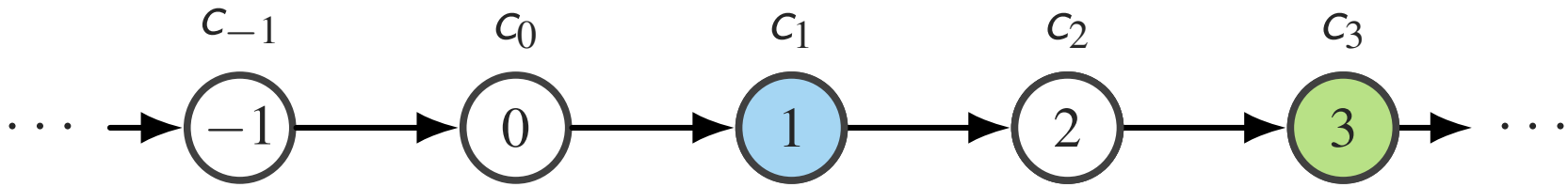
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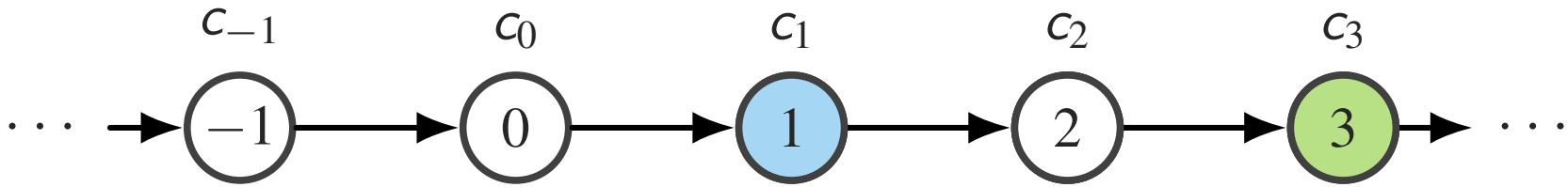
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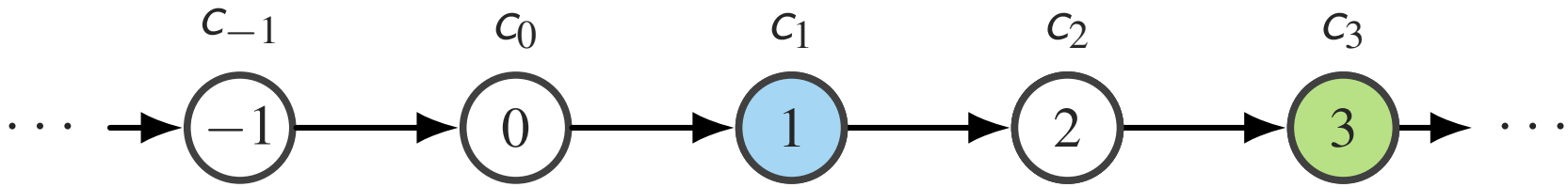


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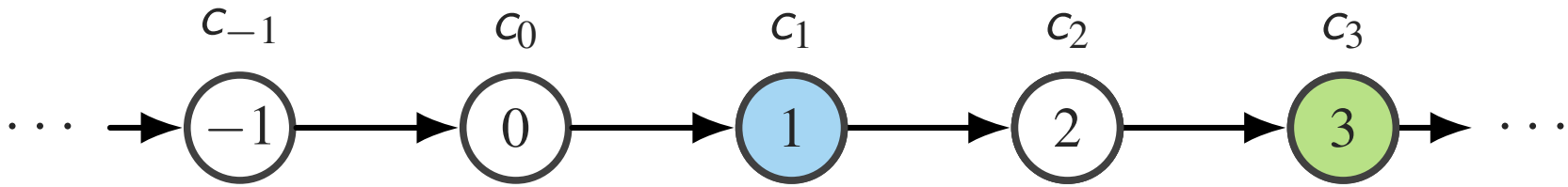
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We construct a "nice" 1-dependent q-coloring with $q = 2 \cdot 10^4$.

An Operator Formulation

We want some Hilbert space H and operators

$$T_1, T_2, \ldots, T_q : H \to H$$

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(T3) $p(a_1 a_2 \ldots a_k) \geq 0$ for all finite words

Enforcing T1: $T_a^2 = 0$

The idea. Having *positive* and *negative* colors.

Let $s = 10^4$ and $q = 2s$.

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Contraction Lemma. For all X, we have that

$$\|X - (2/s)S(X)\| \leq \lambda \|X\| \quad \text{for} \quad \lambda = 3/4 + O(s^{-1/2}) < 1$$

more on this next week!

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The canonical orthogonal basis of H corresponds to *unsigned* coloring of the path (i.e., proper s-colorings):

$$|e\rangle$$
$$|1\rangle, |2\rangle, |3\rangle, |4\rangle, \ldots, |s\rangle$$
$$|12\rangle, |13\rangle, |14\rangle, \ldots |21\rangle, |23\rangle, \ldots |31\rangle, \ldots$$
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$$V_n = \sum_{g,h \in G} \gamma_{g,h} |g\rangle\langle h| \quad \text{is a } \textit{finite} \text{ sum}$$

Warning: Group Algebra Incoming

We have an operator

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Interpret the operator as a polynomial

$$V_n' = \sum_{g,h \in G} \gamma_{g,h} [g,h] \in \mathbb{C}[G \times G]$$

with infinitely many indeterminates $[g,h]$ for $g,h \in G$.

we want an operator in $\mathsf{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$

T1/T2/T3 in $\mathbb{C}[G \times G]$?

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If $V'_n = W^\dagger W$ (wrt multiplication in $\mathbb{C}[G \times G]$) then the "corresponding finite matrix" is always PSD

Can we find some $W = W^\dagger$ such that $W^2 \approx V'_n$?
And use $K_a W^2 K_a$

Taylor Approximation

$(s/2)V_n' = I + R_n$ where $\|R_n\| = O(s^{-1/2})$.

Use

$$\sqrt{I + R_n} = I + R_n/2 - R_n^2/8 + R_n^3/16 + \dots$$

Take $X = I + R_n/2 - R_n^2/8 + \dots$ a finite truncation so that

$$\sum_a K_a W^2 K_a \approx I \quad \text{where} \quad W = \sqrt{\frac{2}{s}} X$$

Summary

Theorem. For all $\delta > 0$, there is a self-adjoint $W \in \mathbb{C}[G \times G]$ s.t.

$$\left\| \sum_a K_a W^2 K_a - I \right\| \leq \delta$$

Proof Sketch.

1. Get $\sum_a P_a V_n P_{-a} \approx |e\rangle\langle e|$
2. Vectorize from operators to group algebra
3. Get W such that $W^2 \approx V_n$ using Taylor approximation

Remarks

- The P_a and K_a projections ensure the coloring is proper
- Using W^2 ensures positivity

The Technical Challenges

Proving the **Contraction Lemma.** For all X, we have that

$$\|X - (2/s)S(X)\| \leq \lambda \|X\| \quad \text{for} \quad \lambda = 3/4 + O(s^{-1/2}) < 1$$

For the Taylor series to converge, need a universal constant C st

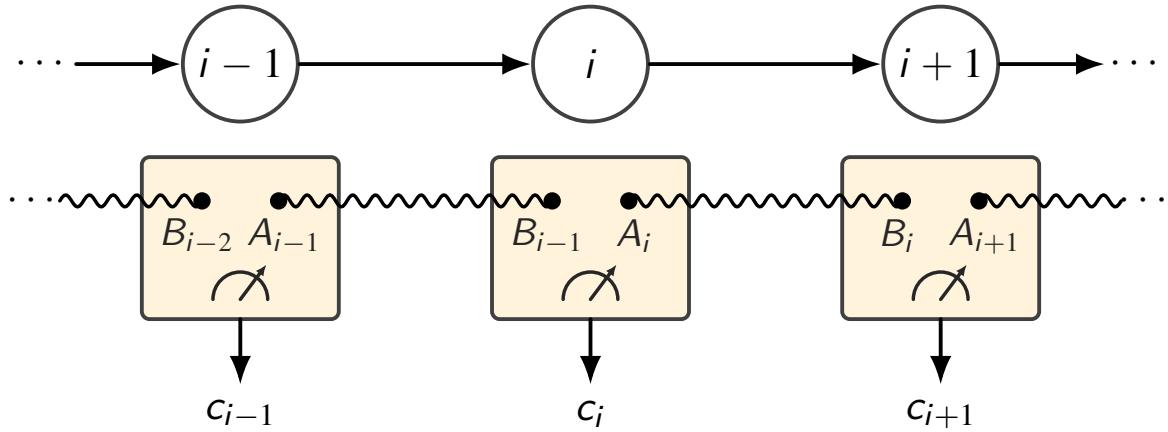
$$\|R_n\| < 1/C \quad \text{and} \quad \underbrace{\|A \cdot B\|}_{\text{mult. in } \mathbb{C}[G \times G]} \leq C\|A\|\|B\|$$

More on this next week!

Part II:

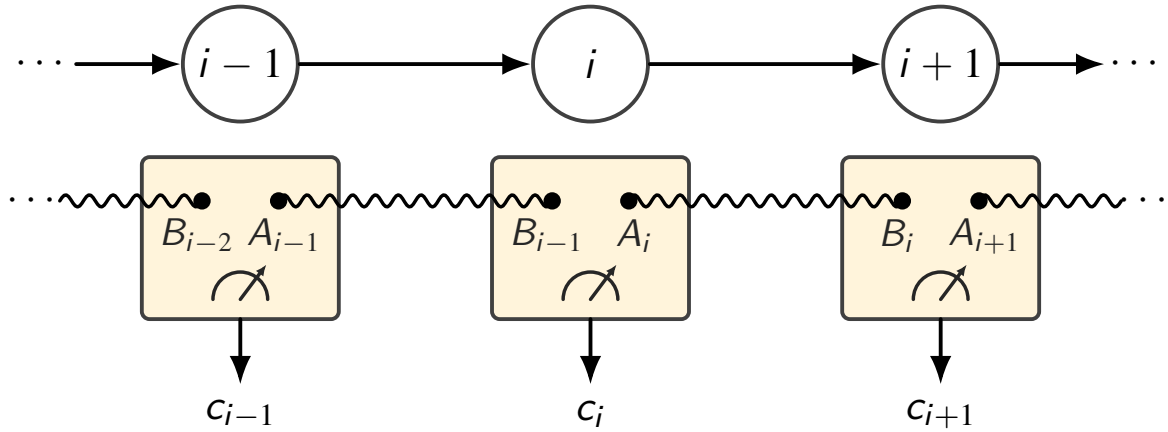
The Finite Dimensional Realization

The Quantum Essential



The state is $|\Phi_N\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |ii\rangle$

The Quantum Essential

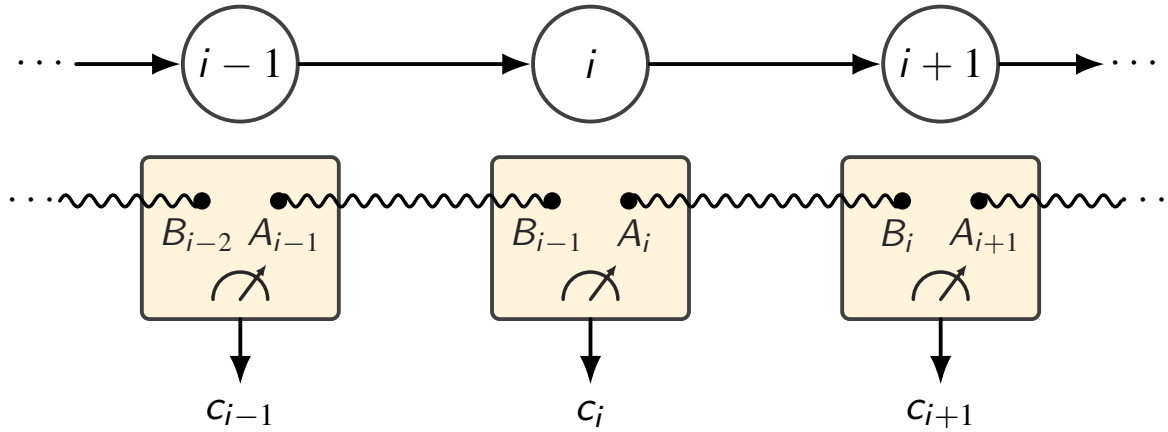


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A measurement with q outcomes is described by a POVM:

- PSD matrices $M_1, M_2, \ldots, M_q \in \mathsf{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$
- $\sum_a M_a = I$ (probabilities sum to one)

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$$\Pr[c_i = c_{i+1} = a] = \frac{1}{N^2} \operatorname{Tr}((M_a \otimes M_a)(I_N \otimes |\Phi_N\rangle\langle\Phi_N| \otimes I_N))$$

Energy: A key insight from Gur-Li

Def. For a PSD matrix $M \in L(\mathcal{A} \otimes \mathcal{B})$, let its energy be

$$\mathcal{E}(M) = \mathrm{Tr}\Big(\mathrm{Tr}_{\mathcal{A}}(M)^T \, Tr_{\mathcal{B}}(M)\Big)$$

Key fact. Energy captures error probabilities

$$\begin{aligned}\Pr[c_i = c_{i+1} = a] &= \frac{1}{N^2}\,\mathrm{Tr}((M_a \otimes M_a)(I_N \otimes |\Phi_N\rangle\langle\Phi_N| \otimes I_N)) \\ &= \frac{1}{N^3}\mathcal{E}(M_a)\end{aligned}$$

All you need to know today. The following are equivalent

1. M has zero-energy
2. there exists $U \oplus \overline{U^{\perp}} = \mathbb{C}^N$ such that $\mathrm{supp}(M) \subseteq U \otimes \overline{U^{\perp}}$.

From $\mathbb{C}[G \times G]$ to $\mathsf{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$

$$[g,h] \overset{\pi}{\mapsto} U_g \otimes \overline{U_h}$$

$$\sum_{g,h} \gamma_{g,h}[g,h] \overset{\pi}{\mapsto} \sum_{g,h} \gamma_{g,h}(U_g \otimes \overline{U_h})$$

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We need this map π to preserve the structure of $\mathbb{C}[G \times G]$

$$[g,h] \cdot [g',h'] \overset{\pi}{\mapsto} (U_g \otimes \overline{U_h})(U_{g'} \otimes \overline{U_{h'}})$$

$$=$$

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How? Take matrices $U_1, U_2, \ldots, U_s$ for each generator of G s.t.

$$U_i^\dagger = U_i \quad \text{and} \quad U_i^2 = I_N$$

and let $U_g = U_{i_1} U_{i_2} \ldots U_{i_k}$ for $g = i_1 i_2 \ldots i_k$.

The Infinite Construction gives Zero Energy PSD

In Part I, we constructed objects

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Problem. The M_a do **not** sum to I_{N^2}

Summing to one

Theorem. For all $\delta > 0$, there is a self-adjoint $W \in \mathbb{C}[G \times G]$ s.t.

$$\left\| \sum_a K_a W^2 K_a - [e, e] \right\| \leq \delta \, .$$

Problems.

1. Need to preserve the norm, approximately at least
2. Need to normalize the resulting POVMs
   without gaining too much energy

Asymptotic Freeness

What we construct. Matrices $U_1, \ldots, U_s$ such that

$$\frac{1}{N} \operatorname{Tr}\left(U_g^\dagger U_h\right) \longrightarrow 1(g = h) \quad \text{as} \quad N \to \infty$$

Lemma. For $Z \in \mathbb{C}[G \times G]$,

$$\frac{1}{N}\|\pi(Z)\|_F \longrightarrow \|Z\| \quad \text{as} \quad N \to \infty$$

More on this during Longcheng's talk!

We get zero-energy PSD $M_1, M_2, \ldots, M_q \in \mathsf{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$ such that

$$\frac{1}{N}\left\|\sum_a M_a - I\right\|_F \leq \varepsilon$$

Lemma. Given $M_1, M_2, \ldots, M_q$ as above, we can construct PSD $\widetilde{M}_1, \widetilde{M}_2, \ldots, \widetilde{M}_q$ such that

$$\sum_a \widetilde{M}_a = I \quad \text{and} \quad \mathcal{E}(\widetilde{M}_a) \leq O(\varepsilon^2 N^3)$$

More on this during Longcheng's talk!

And that's it! The probability of having a monochromatic edge is

$$\Pr[c_i = c_{i+1} = a] = \frac{1}{N^3}\mathcal{E}(\widetilde{M}_a) \leq O(\varepsilon^2)$$

Concluding Remarks

Summary of the Proof

The Infinite Construction

1. operator formulation
2. existence through fixpoint
3. vectorize
4. taylor approximation

The Finite Construction

1. structure preserving map from infinite to $\mathsf{L}(\mathbb{C}^N \otimes \mathbb{C}^N)$
2. get zero-energy PSD that almost sum to 1
3. renormalize with little energy increase

Conclude. POVM with energy $\to 0$ implies existence of an algorithm where error $\to 0$.

Follow-up work

Can we improve the number of colors? (yes, 25 colors works)

What is the optimal number of colors for 1-way 1-round?

What about two-way communication? What about more rounds?

Can we do the same for other problems?

- $(\Delta + 1)$-coloring
- maximal matching
- sinkless orientation

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Thanks!